A TEXT-BOOK

ON

SPHERICAL TRIGONOMETRY

BY

ROBERT E. MORITZ

PH.D. (NEBRASKA), PH.N.D. (STRASSBURG), PROFESSOR OF MATHEMATICS,
UNIVERSITY OF WASHINGTON

FIRST EDITION

FIRST THOUSAND

NEW YORK
JOHN WILEY & SONS, INC.
LONDON: CHAPMAN & HALL, LIMITED
1913

PREFACE

In preparing this little text the author has followed the general plan adopted in his plane trigonometry. Whatever unusual merit the book possesses must be sought for largely in the following points:

1. Superfluous figures in the answers to problems are suppressed on the ground that the current practice of giving answers to a degree of accuracy not warranted by the data is detrimental in its influence on the student.

2. The first exercises under each case of triangles have the parts given to the nearest minute only. This is done to relieve the student of the task of interpolation until he has acquired some familiarity with his formulas. After that the parts are given to the nearest tenth of a minute and then follow exercises in which the data are expressed to the nearest second.

3. It is believed that a proof of Napier's Rules of Circular Parts appears here for the first time in an elementary textbook.

4. Alternate proofs are given or suggested for all fundamental theorems.

5. The three fundamental relations of the parts of oblique spherical triangles are proven simultaneously by the principles of analytical geometry enabling classes which have some familiarity with analytical geometry to cover the present subject in a minimum of time.

6. More complete lists of applied problems will be found than is customary in the current texts.

The author wishes to acknowledge his indebtedness to his colleague, Professor S. L. Boothroyd, Associate Professor of Astronomy, who has prepared the list of problems from astronomy and has read the entire manuscript.

All references to plane trigonometry are to the author's "Elements of Plane Trigonometry," Wiley and Sons, New York.

ROBERT E. MORITZ.

CONTENTS

CHAPTER I

INTRODUCTION

CHAPTER II

RIGHT AND QUADRANTAL SPHERICAL TRIANGLES

CHAPTER III

PROPERTIES OF OBLIQUE SPHERICAL TRIANGLES

CHAPTER IV

SOLUTION OF OBLIQUE SPHERICAL TRIANGLES

SPHERICAL TRIGONOMETRY

CHAPTER I

INTRODUCTION

1. Definition of Spherical Trigonometry. If three points on any surface are joined by the shortest lines lying in the surface that it is possible to draw between these points a triangle is formed. Every such triangle has six parts, three sides and three angles. In general the sides are not straight lines but geodesic lines, that is, the shortest lines that can be drawn on the surface connecting the points. Thus every class of surfaces gives rise to a special trigonometry whose object is the investigation of the relations between the parts of the triangle and the study of the functions necessary for the determination of the unknown parts of a triangle from a sufficient number of given parts.

If the surface under consideration is the plane, the geodesics are straight lines and the triangles plane triangles, whose properties and those of the functions necessary for their solution have been considered in plane trigonometry. If the points lie on the surface of a sphere the geodesics are arcs of great circles, the triangles are called spherical triangles, and the corresponding trigonometry, spherical trigonometry. Briefly stated,

Spherical Trigonometry *deals with the relations among the six parts of a spherical triangle and the problems which may be solved by means of these relations.* The most important of these problems consist in the computation of the unknown parts of a spherical triangle from three given parts. It will be found that the solution of spherical triangles requires no functions other than those employed in plane trigonometry.

2. The Uses of Spherical Trigonometry. It is obvious that the triangle formed by three points on the earth's surface is not a plane triangle but a spherical triangle, for the distances between are measured not along straight lines but along arcs of great circles. It

is only when the distances are comparatively small that the sides may be considered straight lines and that the formulas of plane trigonometry give fairly approximate results. Hence geodetic surveying, that is surveying on a large scale, requires a knowledge of spherical trigonometry. The same is true of navigation when the bearings and distances of distant ports are under consideration. Strictly speaking since the earth is not a perfect sphere but a spheroid, such problems require a knowledge of spheroidal trigonometry, a branch of trigonometry whose study demands the introduction of functions other than those considered in plane trigonometry, but for many purposes the laws of spherical trigonometry give sufficiently accurate approximations.

While a knowledge of spherical trigonometry is of great importance to the surveyor and navigator, it is of even greater importance to the astronomer. The positions of all heavenly bodies are referred to the surface of an imaginary sphere, the celestial sphere, which encloses them all. In fact it is the dependance of astronomy upon spherical trigonometry that first led to its study by the ancients, long before plane trigonometry was thought of as a separate branch of science. Spherical trigonometry is, as it were, the elder sister of plane trigonometry.

Besides the uses already mentioned, spherical trigonometry furnishes the best possible review and constitutes one of the most interesting applications of the principles of plane trigonometry. Spherical trigonometry embodies the results of plane trigonometry in much the same measure that solid geometry embodies the results of plane geometry.

Finally, spherical trigonometry is worthy of study for its own sake because of the marvellous relations which it reveals and the simplicity, elegance, and beauty of the formulas in which its results are embodied.

3. Spherical Trigonometry Dependent on Solid Geometry. Just as plane trigonometry presupposes a certain knowledge of plane geometry so spherical trigonometry requires an acquaintance with solid geometry, especially with that portion of it which deals with the sphere. The student should, therefore, have a textbook on solid geometry ready at hand while pursuing this study in order to familiarize himself anew with the theorems and definitions which are presupposed in the discussions which follow. He should also provide himself with a small wooden or plaster of paris sphere and construct

his figures on it whenever he has difficulty in visualizing the figures called for in his study.

4. Classification of Spherical Triangles. Like plane triangles, spherical triangles are classified in two ways: first, with reference to the sides and second, with reference to the angles.

A spherical triangle is said to be *equilateral*, *isosceles*, or *scalene*, according as it has three, two, or no equal sides. Since each side of a

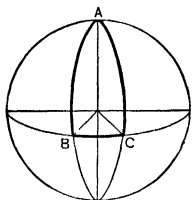



Fig. 1.

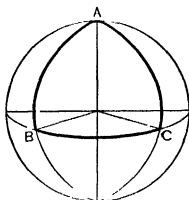



Fig. 2.

spherical triangle may have any value less than 180°,* one, two, or all three of the sides may be quadrants. If one side is a quadrant, the triangle is called *quadrantal*, if two, *biquadrantal*, if all three, *triquadrantal*.

A *right spherical triangle* is one which has a right angle; an *oblique spherical triangle* is one which has none of its angles a right angle.

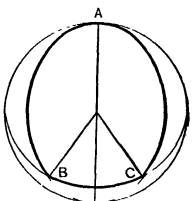



Fig. 3.

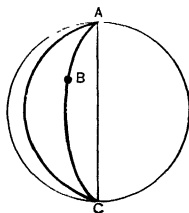



Fig. 4.

Oblique spherical triangles are *obtuse* or *acute* according as they have or have not an obtuse angle. Since the sum of the angles of a spherical

* By the number of degrees in an arc we mean, of course, the number of degrees in the angle which the arc subtends at the center of the sphere. The number of degrees in an arc being given, the length of the arc is at once found from the relation, $s = r\theta$, where r is the radius of the sphere and θ the radian measure of the angle. (See Pl. Trig., Art. 90.)

triangle may have any value between 180° and 540° and no single angle can exceed 180°, a spherical triangle may have two or even three right angles. If it has two right angles it is called ***birectangular*** (Fig. 1), if three, ***trirectangular*** (Fig. 2). For the same reason a spherical triangle may have two or even three obtuse angles (Fig. 3).

If two points on a sphere are at the extremities of the same diameter any great circle passing through one of the points will pass also through the other. Two such points, therefore, cannot be the vertices of a spherical triangle, for the great circles connecting these points with any third point will coincide and the resulting figure will not be a triangle but a lune (Fig. 4).

5. Co-lunar Triangles. If the arcs AB, AC (Fig. 5) forming two sides of any spherical triangle be produced, they will meet again in some point A', forming a lune. The third side BC divides this lune into two triangles, the original triangle ABC, and the triangle $A'BC$. The triangle $A'BC$ thus formed is said to be *co-lunar* with the triangle ABC. It is obvious that any given triangle has three co-lunar triangles, one corresponding to each angle of the triangle. Thus the triangle ABC (Fig. 5) has the three co-lunar triangles $A'BC$, $AB'C$, ABC', where A', B', C' are the opposite poles of the vertices A, B, C of the triangle ABC.

Fig. 5.

Since the angles of a lune are equal, and the sides of the lune semicircles, it follows that the parts of the co-lunar triangles may be immediately expressed in terms of the parts of the original triangle. If we denote the sides of the triangle ABC by a, b, c, and the angles by A, B, C, the corresponding parts of the co-lunar triangles are as follows:

Triangle.	Sides.			Angles.		
ABC	a	b	c	A	B	C
$A'BC$	a	$180°-b$	$180°-c$	A	$180°-B$	$180°-C$
$AB'C$	$180°-a$	b	$180°-c$	$180°-A$	B	$180°-C$
ABC'	$180°-a$	$180°-b$	c	$180°-A$	$180°-B$	C

6. Use of Co-lunar Triangles. Any general formula expressing a relation between the parts of a spherical triangle must continue true when applied to the co-lunar triangles. We may, therefore, substitute in any such formula for any two sides and their opposite

angles their supplements, leaving the third side and angle unchanged. This process frequently leads to new relations among the parts of the triangle.

Thus, after it has been shown that for any triangle

$$\cos\frac{a-b}{2}\cos\frac{C}{2} = \sin\frac{A+B}{2}\cos\frac{c}{2},$$

we obtain, by applying this formula to the co-lunar triangle $A'BC$,

$$\cos\frac{a-(180^\circ-b)}{2}\cos\frac{180^\circ-C}{2} = \sin\frac{A+(180^\circ-B)}{2}\cos\frac{180^\circ-c}{2},$$

which reduces to the new formula

$$\sin\frac{a+b}{2}\sin\frac{C}{2} = \cos\frac{A-B}{2}\sin\frac{c}{2}.$$

Exercise 1

1. Show that every birectangular spherical triangle is also biquadrantal, and every trirectangular triangle is also triquadrantal.

2. Prove the converse of the proposition in Problem 1.

3. The co-lunar triangles of any right spherical triangle are right spherical triangles, and the co-lunar triangles of any quadrantal triangle are quadrantal.

4. The co-lunar triangles of an equilateral spherical triangle are isosceles.

5. It will be shown later that for any spherical triangle

$$\cos\frac{a+b}{2}\sin\frac{C}{2} = \cos\frac{A+B}{2}\cos\frac{c}{2}.$$

By applying this formula to the co-lunar triangle $A'BC$ show that

$$\sin\frac{a-b}{2}\cos\frac{C}{2} = \sin\frac{A-B}{2}\sin\frac{c}{2}.$$

6. It will be shown later that for any spherical triangle

$$\sin\frac{C}{2} = \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a \sin b}},$$

where $$s = \frac{a+b+c}{2}.$$

By applying this formula to the co-lunar triangle ABC' show that

$$\cos\frac{C}{2} = \sqrt{\frac{\sin s \sin(s-c)}{\sin a \sin b}}.$$

7. In Fig. 6, ABC is any right spherical triangle, right-angled at 1. With B as a pole construct a great circle cutting CB produced in 2 and BA produced in 3. With A as a pole construct a great circle cutting AB produced in 4 and CA produced in 5. The resulting figure is a curvilinear pentagon bordered by five spherical triangles. Show that each of these triangles is right-angled and determine all their parts as indicated in the figure. (Remark. The dashes over the letters indicate complements, thus $\bar{A} = 90^\circ - A$, $\bar{a} = 90^\circ - a$, $\bar{c} = 90^\circ - c$, etc.)

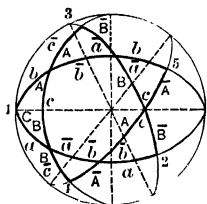



Fig. 6.

7. Polar Triangles. If from the vertices of any spherical triangle ABC as poles, great circles are drawn they will divide the surface of the sphere into eight associated spherical triangles *one of which* is called the *Polar* of the triangle ABC, and is determined as follows:

The great circles whose poles are B and C respectively intersect in two points which lie on opposite sides of the arc BC. Let A' be that one of these two points which lies on the same side of BC as A. The great circles whose poles are C and A respectively intersect in two points which lie on opposite sides of the arc CA. Let B' be that one of the two points which lies on the same side of CA as B. Similarly, let C' be that one of the intersection points of the great circles whose poles are A and B, respectively, which lies on the same side of the arc AB as the vertex C. The triangle whose vertices are A', B', C' is the polar of the triangle ABC.

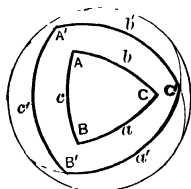
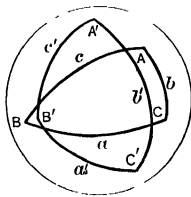
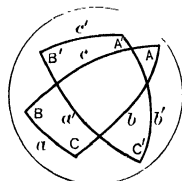



Fig. 7.

Fig 8.

Fig. 9.

Just as in triangle ABC we use A, B, C to denote the angles and a, b, c to denote the sides opposite these angles, so A', B', C' denote the angles and a', b', c' the sides opposite these angles in the polar

It is necessary to recall the two fundamental properties of polar triangles:

I. *The relation of a triangle to its polar is mutual, that is, if $A'B'C'$ is the polar of ABC then ABC is the polar of $A'B'C'$.* Since each of these triangles is the polar of the other, two such triangles are referred to as *polar triangles.*

II. *In two polar triangles each angle is the supplement of the opposite side in the other, and each side the supplement of the opposite angle in the other.* In symbols,

$$A + a' = 180^\circ, \qquad A' + a = 180^\circ,$$
$$B + b' = 180^\circ, \qquad B' + b = 180^\circ,$$
$$C + c' = 180^\circ, \qquad C' + c = 180^\circ.$$

8. The Six Cases of Spherical Triangles. It will be shown presently that the six parts of any spherical triangle are so related that when any three are given the remaining three can be found. The three given parts may be:

I. *The three sides.*
II. *The three angles.*
III. *Two sides and the included angle.*
IV. *Two angles and the included side.*
V. *Two sides and the angle opposite one of them.*
VI. *Two angles and the side opposite one of them.*

There are six cases of spherical triangles while there are only three cases of plane triangles. This is because Cases IV and VI above reduce to the same case for plane triangles, since any two angles of a triangle determine the third. Also Case II above is ruled out for plane triangles since the three angles of a plane triangle determine only the shape but not the magnitude of the triangle.

9. Solution of Spherical Triangles. There are two distinct methods of finding the unknown parts of a spherical triangle from three known parts:

I. *The Graphic Method.* This consists of actually constructing the triangle on a material sphere. The unknown parts may then be found by measurement.

II. *The Method of Spherical Trigonometry.* The unknown parts are obtained by computation by means of formulas expressing the relation of the unknown parts to the parts which are given.

The first method is purely geometrical and is subject to all the errors of construction and inaccuracies of measurement pointed out in Pl. Trig., Art. 3. It is valuable as a rough check on the second method rather than as an independent method of solution.

The second method gives the unknown parts to a degree of accuracy limited only by the accuracy of the data and the number of places of the tables employed in the computation. This is the method employed in Geodesy, in Astronomy, and whenever precision is necessary or desirable. The derivation of the formulas employed by the second method and their application to the solution of the six cases of triangles constitutes an important part of Spherical Trigonometry.

10. The Use of the Polar Triangle. By the use of the polar triangle the second, fourth, and sixth case of spherical triangles may be made to depend on the first, third, and fifth respectively. Consider for instance Case II, in which the three angles are given. From the relations of Art. 7 the three sides of the polar triangle are known, this triangle may, therefore, be solved by Case I, and having found the angles of this triangle, the sides of the original triangle are given by the relations of Art. 7. Similarly, Case IV may be solved by Case III, and Case VI by Case V.

Again by means of the polar triangle any known relation between the parts of a triangle may be made to yield another relation, which frequently turns out to be new; for a relation which holds for every triangle must remain true when applied to the polar, that is, it must hold true if we put for each side the supplement of the opposite angle and for each angle the supplement of the opposite side. Thus if in the formula

$$\cos \tfrac{1}{2}(a-b)\cos \tfrac{1}{2}C = \sin \tfrac{1}{2}(A+B)\cos \tfrac{1}{2}c$$

of Art. 6 we put

$$a = 180^\circ - A',\ b = 180^\circ - B',\ C = 180^\circ - c',$$
$$A = 180^\circ - a',\ B = 180^\circ - b',\ c = 180^\circ - C',$$

we obtain

$$\cos\frac{(180^\circ - A') - (180^\circ - B')}{2}\cos\frac{180^\circ - c'}{2} =$$
$$\sin\frac{(180^\circ - a') + (180^\circ - b')}{2}\cos\frac{180^\circ - C'}{2},$$

which on reducing becomes

$$\cos \tfrac{1}{2} (A' - B') \sin \tfrac{1}{2} c' = \sin \tfrac{1}{2} (a' + b') \sin \tfrac{1}{2} C',$$

or dropping accents

$$\cos \tfrac{1}{2} (A - B) \sin \tfrac{1}{2} c = \sin \tfrac{1}{2} (a + b) \sin \tfrac{1}{2} C.$$

11. Construction of Spherical Triangles.

Case I. *Given the three sides, a, b, c.*

On a sphere lay off an arc BC equal to a.* With B as a pole and an arc equal to c draw a small circle and with C as a pole and an arc equal to b draw another small circle. Either of the intersection points, A, A', of these small circles will be the vertex of a triangle whose other vertices are B and C and whose sides are the three given parts, a, b, c.

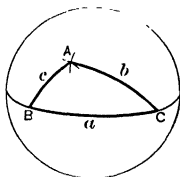



Fig. 10.

Case II. *Given the three angles, A, B, C.*

By Case I construct the polar triangle whose sides are

$$a = 180^\circ - A, \quad b = 180^\circ - B, \quad c = 180^\circ - C.$$

The polar of this triangle will be the required triangle.

Case III. *Given two sides and the included angle, a, b, C.*

On a sphere draw an arc CM of a great circle and on it lay off an arc CB equal to a. Through C draw an arc CN making an angle C with CM.† On CN lay off an arc CA equal to b and join A and B by an arc of a great circle. Then ABC will be the required triangle.

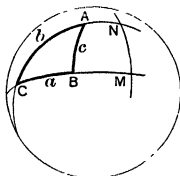



Fig. 11.

Case IV. *Given two angles and the included side, A, B, c.*

By Case III construct the polar triangle whose two sides and included angle are:

$$a = 180^\circ - A, \quad b = 180^\circ - B, \quad C = 180^\circ - c.$$

The polar of this triangle will be the required triangle.

* To lay off an arc equal to a means to lay off an arc of a great circle containing a degrees. This may be readily done by means of a strip of paper or cardboard equal in length to a semicircumference of the sphere and dividing it into 180 equal divisions. Each division will then represent one degree of angular measure on the sphere.

† This is most easily done as follows: From C as a pole draw the arc of a great circle. Let M be its intersection with CM. On this arc lay off MN equal to C. Join N and C by an arc of a great circle. Then NCM will be the required angle. (Why?)

Case V. *Given two sides and the angle opposite one of these sides,* a, b, A.

We distinguish two cases according as the angle A is acute or obtuse.

I. *A acute.*

On a sphere (Fig. 12) draw two arcs, AM and AN, making an angle A with each other and let A and A' be their points of intersection. On one of these arcs, as AN, lay off AC equal to b. With C as a pole and an arc equal to a describe a small circle.* In general this circle will intersect the arc AM in two points, B and B', either of which, if its angular distance from A is less than 180°, will form the third vertex of a triangle whose other two vertices are A and C and which will contain the three given parts.

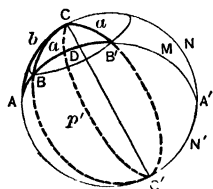



Fig. 12.

Let $p = CD$ be the arc through C which is perpendicular to AM.

(*a*) If a is less than p, the small circle will not intersect AM and no triangle exists having the given parts. The solution is impossible.

(*b*) If $a = p$, there is one solution. The resulting triangle has a right angle at D.

(*c*) If a is greater than p but less than the shorter of the two sides, $AC = b, CA' = 180° - b$, there will be two solutions, ACB and ACB'.

(*d*) If a is greater than the shorter of the two sides b and $180° - b$ but less than the longer, there will be one solution.

(*e*) If a is greater than the longer of the two sides b and $180° - b$ there will be no solution.

II. *A obtuse.*

Draw the two arcs AM and AN' (Fig. 12), making the angle A with each other. On one of these arcs, as AN', lay off AC' equal to b. With C' as a pole and an arc equal to a describe a small circle which, in general, will intersect the arc AM in two points, B and B', either of which, if its angular distance from A is less than 180°, will form the third vertex of a triangle whose other two vertices are A and C'.

Let $p' = C'D$ be the arc through C' which is perpendicular to AM. As p is the shortest arc that can be drawn from C to AM, so p' is the longest arc that can be drawn from C' to AM.

* This may be done by means of a pair of compasses.

(*a*) If a is greater than p', the small circle will not intersect AM and no triangle exists having the given parts. There is no solution.

(*b*) If $a = p'$, there is one solution. The resulting triangle has a right angle at D.

(*c*) If a is less than p' but greater than the longer of the two sides, $AC' = b$, $C'A' = 180° - b$, there will be two solutions, $AC'B$ and $AC'B'$.

(*d*) If a is less than the longer of the two sides, b and $180° - b$, but greater than the shorter, there will be one solution.

(*e*) If a is less than the shorter of the two sides, b and $180° - b$, there will be no solution.

Case VI. *Given two angles and the side opposite one of them*, A, B, c.

By Case V construct the polar triangle whose parts are $a = 180° - A$, $b = 180° - B$, $A = 180° - a$. The polar of this triangle will be the required triangle. As in Case V, so here there may be either one or two solutions or the solution may be impossible.

12. The General Spherical Triangle. We have defined a spherical triangle as the figure formed by joining three points on a sphere, which lie not in the same great circle, and no two of which are opposite ends of the same diameter, by the *shortest* great arcs. From this it follows that each side of a spherical triangle is less than a semicircumference, and its angular measure less than 180°.

Now the great circle drawn through two points is divided by those points into two arcs either of which may be considered the arc between the two points. If one of these arcs is less than 180° the other will be greater than 180° for their sum is always 360°. Hence if we drop the word *shortest* from the above definition, the resulting definition admits triangles whose sides have any value between 0° and 360°. Such triangles are called *general spherical triangles*. Since the arc between each two vertices may be chosen in two ways there are eight general triangles having the same three vertices. Fig. 13 shows two of these triangles, the triangle $AMBC$ and the triangle $AM'BC$.

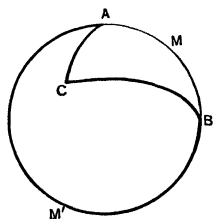



Fig. 13.

The study of general spherical triangles forms the object of Higher Spherical Trigonometry. Their principal applications are found in astronomy where it is frequently necessary to consider triangles

whose sides or angles exceed 180°. *We observe that every spherical triangle, one or more of whose parts exceed 180°, may be solved by means of another whose parts are less than 180°, though this is not the simplest way of treating such triangles.* In the present text we shall limit our discussion to triangles which satisfy the first definition, that is, triangles each of whose parts is less than 180°.

Exercise 2

1. Prove the two theorems of Art. 7.

2. Prove that the polar of a right spherical triangle is quadrantal, and conversely, that the polar of a quadrantal triangle is a right triangle.

3. Prove that the polar of a birectangular spherical triangle is biquadrantal, and conversely, that the polar of a biquadrantal triangle is birectangular.

4. Prove that a trirectangular triangle is its own polar.

5. If the sides of a triangle are each less than 90° it lies wholly within its polar; if each of its sides is greater than 90° its polar lies wholly within it.

6. In any spherical triangle $a + b + c < 360°$. By applying this relation to the polar show that in every spherical triangle

$$180° < A + B + C < 540°.$$

7. In every spherical triangle the sum of two sides is greater than the third side, that is $a + b > c$. By applying this relation to the polar show that in every spherical triangle the difference between any angle and the sum of the other two is less than 180°, that is, $A + B - C < 180°$.

8. It will be shown later that in any spherical triangle

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

By applying this formula to the polar triangle show that also

$$\cos A = - \cos B \cos C + \sin B \sin C \cos a.$$

9. By applying the formulas of Problem 6, Exercise I, to the polar triangle, deduce the two new formulas,

$$\cos \frac{c}{2} = \sqrt{\frac{\cos (S - A) \cos (S - B)}{\sin A \sin B}}, \quad \sin \frac{c}{2} = \sqrt{- \frac{\cos S \cos (S - C)}{\sin A \sin B}},$$

where

$$S = \frac{A + B + C}{2}.$$

10. Construct the triangle called for in Case IV, Art. 11, without employing the polar triangle.

11. In Case V, Art. 11, write out the conditions under which the construction admits (*a*) one solution, (*b*) two solutions, (*c*) no solution.

CHAPTER II

RIGHT AND QUADRANTAL SPHERICAL TRIANGLES*

13. Formulas for Right Spherical Triangles. Every right triangle has a right angle and five other parts which, beginning with a side including the right angle, are denoted in order by a, B, c, A, b. We shall show that every three of these five parts are so related that when two are given the third may be found. Now the above five parts admit of ten different sets of three, namely:

$$A,\ a,\ c;\quad A,\ b,\ c;\quad A,\ a,\ b;\quad A,\ B,\ b;\quad c,\ a,\ b;$$
$$B,\ b,\ c;\quad B,\ a,\ c;\quad B,\ b,\ a;\quad B,\ A,\ a;\quad c,\ A,\ B;$$

hence we shall find ten formulas for the right spherical triangle.

Let ABC, Fig. 14, be a right spherical triangle, C the right angle. Let O be the center of the sphere and $O - ABC$ the trihedral angle formed by the planes of the great circles whose arcs are a, b, c, respectively. It is shown in geometry that the face angles BOC, COA, AOB are measured by the arcs a, b, c, respectively, and that the dihedral angles OA, OB, OC are equal to the angles A, B, C, respectively. From any point P in OB draw PR perpendicular to OC, and from R draw RS perpendicular to OA. Join P and S. Then SR is perpendicular to PR (why?), and PS is perpendicular to OA (why?). Hence

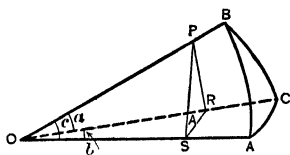



Fig. 14.

triangle ORP has a right angle at R,
triangle OSR has a right angle at S,
triangle OSP has a right angle at S,
triangle PRS has a right angle at R,
and angle PSR equals angle A (why?).†

* If the class has some knowledge of analytical geometry and the teacher wishes to cover the subject in the least time possible, he may omit the work to Art. 26. The fundamental relations for the oblique triangle as there developed may be specialized for the right triangle by putting $C = 90°$.

† See footnote on page 15.

In triangle PRS

$$\sin A = \frac{RP}{SP} = \frac{RP/OP}{SP/OP} = \frac{\sin ROP}{\sin SOP},$$

or $$\sin A = \sin a/\sin c. \tag{1}$$

Interchanging letters $$\sin B = \sin b/\sin c. \tag{2}$$

$$\cos A = \frac{SR}{SP} = \frac{SR/OS}{SP/OS} = \frac{\tan ROS}{\tan POS},$$

or $$\cos A = \tan b/\tan c. \tag{3}$$

Interchanging letters $$\cos B = \tan a/\tan c. \tag{4}$$

$$\tan A = \frac{RP}{SR} = \frac{RP/OR}{SR/OR} = \frac{\tan POR}{\sin ROS},$$

or $$\tan A = \tan a/\sin b. \tag{5}$$

Interchanging letters $$\tan B = \tan b/\sin a. \tag{6}$$

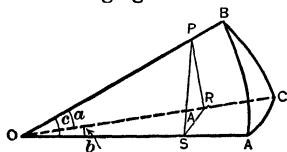



Fig. 14.

$$\sin A = \frac{RP}{SP} = \frac{RP}{OR} \cdot \frac{OR}{OS} \cdot \frac{OS}{SP}$$
$$= \tan POR \cdot \sec ROS \cdot \cot POS$$
$$= \frac{\tan a}{\tan c} \cdot \frac{1}{\cos b},$$

whence, substituting the value of $\tan a/\tan c$ from (4), we have

$$\sin A = \cos B/\cos b. \tag{7}$$

Interchanging letters $$\sin B = \cos A/\cos a. \tag{8}$$

Once more, $$\cos c = \frac{OS}{OP} = \frac{OR}{OP} \cdot \frac{OS}{OR} = \cos POR \cdot \cos ROS,$$

or $$\cos c = \cos a \cos b. \tag{9}$$†

Finally, substituting in (9) for $\cos a$ and $\cos b$ their values from (7) and (8), we obtain

$$\cos c = \cot A \cot B. \tag{10}$$

† Let the student who has undue difficulty in perceiving these relations construct the trihedral angle and the corresponding spherical triangle as follows: From a piece of cardboard or tin cut out a circle with any radius.

Draw four radii OA, OC, OB, OA', making the angles 50°, 70°, 77° 18′, respectively. Cut the circle along the radii OA and OA', and remove the sector AMA'. Cut the remaining sector partly through along OC and OB and bend the cardboard along these radii

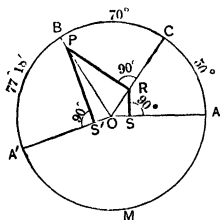



Fig. 15.

14. Plane and Spherical Right Triangle Formulas Compared. The student will be assisted in remembering the ten formulas of the preceding article if he associates them with the corresponding formulas for the plane right triangle, as shown in the following table:

Plane Right Triangles.		Spherical Right Triangles.	
$\sin A = \frac{a}{c}$	$\sin B = \frac{b}{c}$	$\sin A = \frac{\sin a}{\sin c}$	$\sin B = \frac{\sin b}{\sin c}$
$\cos A = \frac{b}{c}$	$\cos B = \frac{a}{c}$	$\cos A = \frac{\tan b}{\tan c}$	$\cos B = \frac{\tan a}{\tan c}$
$\tan A = \frac{a}{b}$	$\tan B = \frac{b}{a}$	$\tan A = \frac{\tan a}{\sin b}$	$\tan B = \frac{\tan b}{\sin a}$
$\sin A = \cos B$	$\sin B = \cos A$	$\sin A = \frac{\cos B}{\cos b}$	$\sin B = \frac{\cos A}{\cos a}$
$c^2 = a^2 + b^2$		$\cos c = \cos a \cos b$	
$1 = \cot A \cot B$		$\cos c = \cot A \cot B$	

15. Generalization of the Right Triangle Formulas. In Fig. 14 the sides a and b are each less than 90°. It remains to show that the formulas in Art. 13 hold for *all* possible values of a and b.

I. *One side adjacent to the right angle greater than 90° and the other less than 90°.*

In the right triangle ABC (Fig. 16), let a be greater than 90° and b less than 90°. The co-lunar triangle $AB'C$ will have a right angle

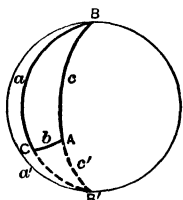



Fig. 16.

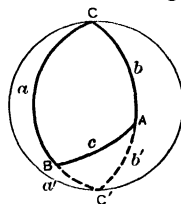



Fig. 17.

at C and the adjacent sides b and $a' = 180° - a$, each less than 90°. We may, therefore, apply the formulas of Art. 13 to this triangle. Thus (1) gives

$$\sin CAB' = \frac{\sin a'}{\sin c'} = \frac{\sin (180° - a)}{\sin (180° - c)} = \frac{\sin a}{\sin c}, \text{ or } \sin A = \frac{\sin a}{\sin c},$$

that is (1) remains true for the triangle ABC.

until OA' meets OA. The figure thus formed will be a right trihedral angle, ABC will form a right spherical triangle, and the lines PR, RS and PS' will form the triangle PRS of Fig. 14.

Similarly each of the other nine formulas will be found true for the triangle ABC.

II. *Each of the sides adjacent to the right angle greater than 90°.*

In the right triangle ABC (Fig. 17), let a and b be each greater than 90°. The co-lunar triangle ABC' will have a right angle at C' and the adjacent sides $a' = 180° - a$ and $b' = 180° - b$, each less than 90°. We may, therefore, apply the formulas of Art. 13 to this triangle. Thus (1) gives

$$\sin BAC' = \frac{\sin a'}{\sin c} = \frac{\sin (180° - a)}{\sin c} = \frac{\sin a}{\sin c}, \text{ or } \sin A = \frac{\sin a}{\sin c},$$

that is (1) holds true for triangle ABC, and similarly each of the other nine formulas will be found true for this case.

This proves that the formulas of Art. 13 may be applied to the solution of every possible right spherical triangle.

16. Napier's Rules of Circular Parts.* Lord Napier, the inventor of logarithms, first succeeded in expressing the ten right triangle formulas by two simple rules. Let us put

$$90° - A = \bar{A}, \quad 90° - c = \bar{c}, \quad 90° - B = \bar{B},$$

then

$$\sin A = \cos \bar{A}, \quad \cos A = \sin \bar{A}, \quad \tan A = \cot \bar{A}, \quad \cot A = \tan \bar{A},$$
$$\sin c = \cos \bar{c}, \text{ etc.,} \qquad \sin B = \cos \bar{B}, \text{ etc.}$$

The ten equations of Art. 13 may then be written as follows, the new formulas being numbered as in Art. 13.

$\sin a = \cos \bar{A} \cos \bar{c}$	(1)	$\sin \bar{A} = \tan b \tan \bar{c}$	(3)
$\sin b = \cos \bar{B} \cos \bar{c}$	(2)	$\sin \bar{B} = \tan a \tan \bar{c}$	(4)
$\sin \bar{B} = \cos \bar{A} \cos b$	(7)	$\sin b = \tan a \tan \bar{A}$	(5)
$\sin \bar{A} = \cos \bar{B} \cos a$	(8)	$\sin a = \tan b \tan \bar{B}$	(6)
$\sin \bar{c} = \cos a \cos b$	(9)	$\sin \bar{c} = \tan \bar{A} \tan \bar{B}$	(10)

Let us now arrange the five parts a, $\bar{B}$, $\bar{c}$, $\bar{A}$, b in their order in a circle as in Fig. 18. Any one of these five parts, as a, being chosen as the *middle* part, the two next to it, as b and $\bar{B}$, are called *adjacent* parts and the remaining two parts,

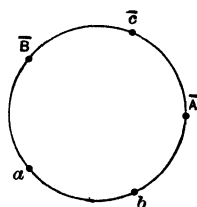



Fig. 18.

* This and the following article may be omitted by those who prefer to memorize the preceding ten formulas as suggested in Art. 14.

as $\bar{A}$ and $\bar{c}$, are called *opposite* parts. Then each of the five equations on the right are contained in

Rule 1. *The sine of the middle part is equal to the product of the* **tangents** *of the* **adjacent** *parts,*

and the five on the left are contained in

Rule 2. *The sine of the middle part is equal to the product of the* **cosines** *of the* **opposite** *parts.*

These two rules are known as *Napier's Rules of the Circular Parts.*

17. Proof of Napier's Rules of Circular Parts. Napier's rules are commonly looked upon as memory rules which happen to include the ten right triangle formulas. They have been proclaimed the happiest example of artificial memory known to man. Because of their supposed artificial character their value as an instrument in mathematics has been questioned. We shall now show that Napier's rules are not mere menotechnic rules but constitute a most remarkable theorem which admits of rigorous proof.

Let ABC_1 be a right spherical triangle, C_1 the right angle. With B as a pole draw a great circle cutting C_1B produced in C_2 and BA produced in C_3. With A as a pole draw a great circle cutting AB produced in C_4 and C_1A produced in C_5. The resulting figure is a spherical pentagon $ABPRS$, bordered by five triangles I, II, III, IV, V.

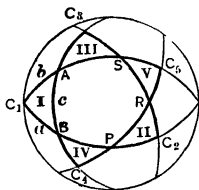



Fig. 19.

Since B is the pole of the arc C_2C_3 the angles at C_2 and C_3 are right angles and since A is the pole of arc C_4C_5 the angles at C_4 and C_5 are right angles. The five triangles are, therefore, right triangles.

Since C_1 and C_2 are right angles, S is the pole of C_1C_2 and consequently SC_1 and SC_2 are quadrants. For like reasons RC_3, RC_4, PC_5, PC_1, BC_2, BC_3, AC_4, AC_5 are quadrants.

With these preliminary observations it is now easy to show that the five triangles I, II, III, IV, V have the same circular parts taken in the same order, while the position of these parts with respect to the right angle is different in each of the triangles.

Let us compare the two triangles ABC_1 and PRC_2 and denote by a_2, B_2, c_2, A_2, b_2 the five parts of II which correspond to a, B, c, A, b of I. Comparing angular measures we have

$$a_2 = C_2R = C_2C_3 - RC_3 = (180^\circ - B) - 90^\circ = 90^\circ - B = \overline{B},$$

$$B_2 = PRC_2 = 180^\circ - PRS = 180^\circ - C_3C_4 = 180^\circ - (C_3B + AC_4 - AB)$$
$$= 180^\circ - (90^\circ + 90^\circ - c) = c,$$

$$c_2 = PR = C_4R + PC_5 - C_4C_5 = 90^\circ + 90^\circ - C_4AC_5$$
$$= 90^\circ + 90^\circ - (180^\circ - A) = A,$$

$$A_2 = RPC_2 = 180^\circ - BPR = 180^\circ - C_1C_5 = 180^\circ - (C_1A + AC_5)$$
$$= 180^\circ - (b + 90^\circ) = \overline{b},$$

$$b_2 = PC_2 = BC_2 - BP = 90^\circ - (C_1P - C_1B) = 90^\circ - (90^\circ - a) = a;$$

hence, $a_2 = \overline{B}$, $B_2 = c$, $C_2 = A$, $A_2 = \overline{b}$, $b_2 = a$.

Now the parts of triangle III may be obtained from those of II, the parts of IV from those of III, and the parts of V from those of IV, in exactly the same way that the parts of II were obtained from those of I. Writing corresponding parts under each other, and remembering that to obtain the circular parts we must replace the hypotenuse and angles of each triangle by their complements, we have the following table:

	Actual Parts	Circular Parts
Triangle I	$a, B, c, A, b,$	$a, \overline{B}, \overline{c}, \overline{A}, b$
Triangle II .	$\overline{B}, c, A, \overline{b}, a$	$\overline{B}, c, \overline{A}, b, a$
Triangle III .	$\overline{c}, A, \overline{b}, \overline{a}, \overline{B}$	$\overline{c}, \overline{A}, b, a, \overline{B}$
Triangle IV	$\overline{A}, \overline{b}, \overline{a}, B, \overline{c}$	$\overline{A}, b, a, \overline{B}, \overline{c}$
Triangle V .	$b, a, B, c, \overline{A}$	$b, a, \overline{B}, \overline{c}, \overline{A}$

The column on the right not only shows that each triangle has the same circular parts taken in the same order, but also that the middle part $\overline{c}$ of the first triangle is successively replaced by $\overline{A}$, b, a, $\overline{B}$ in the other four. Now it was shown in Art. 13 (10), (9), that for the triangle ABC_1,

$$\cos c = \cot A \cot B, \quad \text{or} \quad \sin \overline{c} = \tan \overline{A} \tan \overline{B}, \qquad \text{(I)}$$

$$\cos c = \cos a \cos b, \quad \text{or} \quad \sin \overline{c} = \cos a \cos b, \qquad \text{(II)}$$

hence formulas (I) and (II) hold when any part other than $\overline{c}$ is taken for the middle part, and thus Napier's rules are shown to be necessarily true.

EXERCISE 3

1. Apply the ten formulas for the right spherical triangle to the polar and obtain ten formulas for the quadrantal spherical triangle.

2. Write out the ten equations for the right spherical triangle by means of Napier's rules.

3. From the relation $\cos c = \cos a \cos b$ show that if a right triangle has only one right angle, the three sides are either all acute, or one is acute and the other two obtuse.

4. From the relation $\cos A = \cos a \sin B$ show that the side a is in the same quadrant as the opposite angle A.

5. If in a right spherical triangle $a = c = 90°$, prove that $\cos b = \cos B$.

6. Also if $a = b$, prove that $\cot B = \cos a$.

Prove the following relations for the right triangle ABC:

7. $\cos^2 A - \sin^2 B = - \sin^2 b \sin^2 A$.

8. $\sin A \sin 2b = \sin c \sin 2B$.

9. $\sin^2 a + \sin^2 b - \sin^2 c = \sin^2 a \sin^2 b$.

10. $\sin A \cos c = \cos a \cos B$.

11. $\sin b = \cos c \tan a \tan B$.

12. $\sin^2 A \cos^2 b \sin^2 c = \sin^2 c - \sin^2 b$.

18. To Determine the Quadrant of the Unknown Parts in a Right Spherical Triangle. When an unknown part is found from its cosine, tangent, or cotangent, the sign of the function shows whether the part is in the first or second quadrant, that is, whether it is less than 90° or greater than 90°. In the cases where the unknown part is found from the sine, the following theorems enable us to tell, in every case in which the triangle has but one solution, whether the part is greater or less than 90°.

I. *At least one side of every right spherical triangle is in the first quadrant, the remaining two are either both in the first quadrant or both in the second.* For, since the cosine of an angle in the second quadrant is negative, it is plain that the equation

$$\cos c = \cos a \cos b \qquad \text{(Art. 13 (10))}$$

must have either none or two of the angles a, b, c in the second quadrant.

II. *Either of the oblique angles of a right spherical triangle is in the same quadrant as its opposite side.* For since

$$\sin A = \cos B / \cos b \qquad \text{(Art. 13 (7))}$$

and sin A is always positive, it is plain that cos B and cos b must either be both positive or both negative, that is, B and b and similarly A and a, must be in the same quadrant.

19. The Ambiguous Case of Right Spherical Triangles. When the given parts of a right triangle are an angle and the side opposite, the triangle has two solutions. For, the given parts being A and a (Fig. 20), the co-lunar triangle $A'BC$ as well as the triangle ABC has the given parts. Notice that $A'B$ and $A'C$ are the supplements of AB and AC, respectively, and that angle $A'BC$ is the supplement of angle ABC. Both sets of values are given by the formulas, for, A and a being given, c, b, and B are found from their sines (Art. 13, Equations (1), (5) and (8)).

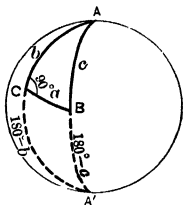



Fig. 20.

20. Solution of Right Spherical Triangles. Napier's rules, or, if it is preferred, the ten formulas in Art. 13, enable us to solve every conceivable right spherical triangle, two parts being given. The procedure in any given case is as follows:

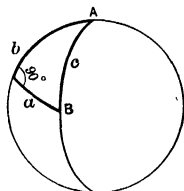



Fig. 21.

I. We consider three parts, two of which are the given parts and the third the part to be found. If these three parts are adjacent we take the middle one for the middle part, if two only are adjacent we take the remaining one for the middle part and by Napier's rules write down the formula relating the three parts.

Thus if A and c are the given parts (Fig. 21), and b is to be found, we take A for the middle part and by Napier's first rule,

$$\sin \bar{A} = \tan b \tan \bar{c}, \text{ that is, } \cos A = \tan b \cot c. \qquad (1)$$

If B is to be found, we take c for the middle part, and again applying Napier's first rule we have

$$\sin \bar{c} = \tan \bar{A} \tan \bar{B}, \text{ that is, } \cos c = \cot A \cot B. \qquad (2)$$

If a is the part required, we take a for the middle part, and applying Napier's second rule, we have

$$\sin a = \cos \bar{A} \cos \bar{c}, \text{ that is, } \sin a = \sin A \sin c. \qquad (3)$$

II. Next we solve the equation for that function which contains the unknown part. Thus to find b, we have from equation (1) above, $\tan b = \cos A \tan c$, to find B we have from (2) $\cot B = \cos c \tan A$, to find a we use equation (3) as it stands.

III. By means of the equations thus obtained and the use of tables we compute the unknown parts, remembering,

(*a*) If the unknown part is found from its cosine, tangent, or cotangent, the algebraic sign of the function determines the quadrant of the angle.

(*b*) If the unknown part is found from its sine, the quadrant of the angle is determined by one of the theorems of Art. 18.

(*c*) If the given parts are an angle and the side opposite, each unknown part has two values which are supplements of each other.

IV. Check. When the unknown parts have been computed, their correctness should be checked by the formula obtained by applying Napier's rules to these parts. Thus in the above example, after b, B, and a have been computed their values must satisfy the formula (a being the middle part)

$$\sin a = \tan \overline{B} \tan b, \text{ that is, } \sin a = \cot B \tan b.$$

EXAMPLE 1.

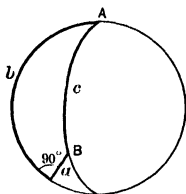
Given
$A = 67° 34' 40''$,
$c = 137° 24' 54''$.

Required
$b = 160° 40' 56''$,
$B = 150° 44' 00''$,
$a = 35° 42' 57''$.

Fig. 22.

Solution.

To find b.

$$\cos A = \cot c \tan b,$$

or,

$$\tan b = \cos A \tan c.$$

$\log \cos A = 9.58141$
$\log \tan c = 9.96334n$*
$\log \tan b = 9.54475n$
$b = 160° 40' 56''$.

To find B.

$$\cos c = \cot A \cos B,$$

or,

$$\cot B = \cos c \tan A.$$

$\log \cos c = 9.86704n$
$\log \tan A = 0.38445$
$\log \cot B = 0.25149n$
$B = 150° 44' 00''$.

To find a.

$$\sin a = \sin A \sin c.$$

$\log \sin A = 9.96586$
$\log \sin c = 9.83038$
$\log \sin a = 9.79624$
$a = 35° 42' 57''$.

Check.

$$\sin a = \cot B \tan b.$$

$\log \cot B = 0.25149n$
$\log \tan b = 9.54475n$
$\log \sin a = 9.79624$ (check).

* n written after a logarithm means that the number of which the logarithm is taken (in this case $\tan c$) has the negative sign.

In this case, since $\tan b$ and $\cot B$ are negative, b and B must be taken in the second quadrant, while a is taken in the first quadrant since by Art. 18 it must be in the same quadrant as the opposite angle A.

EXAMPLE 2.

Given

$$B = 25^\circ\, 36'\, 30'',$$
$$b = 24^\circ\, 20'\, 45''.$$

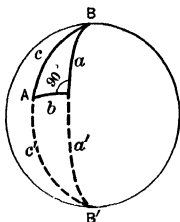



Fig. 23.

Required

$$A = 81^\circ\, 48'\, 30'',$$
$$A' = 98^\circ\, 11'\, 30'',$$
$$c = 72^\circ\, 30'\, 45'',$$
$$c' = 107^\circ\, 29'\, 15'',$$
$$a = 70^\circ\, 44'\, 45'',$$
$$a' = 109^\circ\, 15'\, 15''.$$

Solution.

To find A.

$$\cos B = \sin A \cos b,$$

or,

$$\sin A = \cos B / \cos b.$$

$$\begin{aligned} \log \cos B &= 9.95510 \\ \text{colog} \cos b &= \underline{0.04045} \\ \log \sin A &= 9.99555 \\ A &= 81^\circ\, 48'\, 30''. \\ A' &= 98^\circ\, 11'\, 30''. \end{aligned}$$

To find c.

$$\sin b = \sin B \sin c,$$

or,

$$\sin c = \sin b / \sin B.$$

$$\begin{aligned} \log \sin b &= 9.61515 \\ \text{colog} \sin B &= \underline{0.36430} \\ \log \sin c &= 9.97945 \\ c &= 72^\circ\, 30'\, 45''. \\ c' &= 107^\circ\, 29'\, 15''. \end{aligned}$$

To find a.

$$\sin a = \cot B \tan b.$$

$$\begin{aligned} \log \cot B &= 0.31940 \\ \log \tan b &= \underline{9.65560} \\ \log \sin a &= 9.97500 \\ a &= 70^\circ\, 44'\, 45''. \\ a' &= 109^\circ\, 15'\, 15''. \end{aligned}$$

Check.

$$\sin a = \sin c \sin A.$$

$$\begin{aligned} \log \sin c &= 9.97945 \\ \log \sin A &= \underline{9.99555} \\ \log \sin a &= 9.97500 \text{ (check)}. \end{aligned}$$

In this case there are two solutions. By Art. 18 a and A must be in the same quadrant, hence the acute values of both a and A belong to one triangle and the obtuse values to another. Again, by Art. 18, the three sides a, b, c are either all in the first quadrant, or two are in the second quadrant, hence c is in the same triangle as a, and c' is in the same triangle as a'.

EXERCISE 4

When no answer is given the results must be checked. For the number of significant figures to be retained in the answer see Pl. Trig., Art. 44.

Solve the following right spherical triangles when the parts given are:

1. $a = 81° 25'$, $b = 101° 15'$.
   Ans. $A = 81° 35'$, $B = 101° 08'$, $c = 94° 40'$.
2. $c = 86° 51'$, $B = 18° 04'$.
   Ans. $b = 18° 02'$, $a = 86° 41'$, $A = 88° 58'$.
3. $a = 70° 28'$, $c = 98° 18'$.
   Ans. $A = 72° 15'$, $B = 114° 17'$, $b = 115° 35'$.
4. $c = 118° 40'$, $A = 128° 00'$.
   Ans. $a = 136° 16'$, $b = 48° 24'$, $B = 58° 27'$.
5. $A = 81° 13'$, $B = 65° 24'$.
   Ans. $a = 80° 20'$, $b = 65° 05'$, $c = 85° 56'$.
6. $b = 112° 49'$, $B = 100° 27'$.
   Ans. $a = 26° 00'$, $A = 27° 53'$, $c = 110° 24'$;
   $a' = 154° 00'$, $A' = 152° 07'$, $c' = 69° 36'$.
7. $c = 81° 10'$, $a = 100° 47'$.
8. $A = 75° 23'$, $B = 75° 23'$.
9. $a = 72° 15'$, $B = 83° 25'$.
10. $b = 148° 28'$, $B = 101° 04'$.
11. $a = 43° 40.5'$, $c = 98° 29.1'$.
    Ans. $A = 44° 17.0'$, $B = 98° 11.4'$, $b = 101° 46.3'$.
12. $a = 28° 47.0'$, $b = 110° 27.3'$.
    Ans. $A = 30° 23.1'$, $B = 100° 10.9'$, $c = 107° 50.2'$.
13. $b = 74° 21.9'$, $A = 38° 57.2'$.
    Ans. $B = 80° 14.7'$, $a = 37° 54.1'$, $c = 77° 43.3'$.
14. $A = 49° 15.0'$, $B = 52° 26.0'$.
    Ans. $a = 34° 33.7'$, $b = 36° 24.6'$, $c = 48° 29.3'$.
15. $c = 50° 20.2'$, $A = 101° 29.4'$.
    Ans. $a = 131° 01.7'$, $b = 166° 29.5'$, $B = 162° 20.1'$.
16. $a = 32° 10.8'$, $A = 42° 24.0'$.
    Ans. $b = 43° 34.8'$, $B = 60° 43.2'$, $c = 52° 06.0'$;
    $b' = 136° 25.2'$, $B' = 119° 16.8'$, $c' = 127° 54.0'$.

17. $c = 95° 26.2', b = 12° 37.8'$.
18. $a = 119° 56.1', b = 151° 43.6'$.
19. $A = 70° 56.9', B = 39° 25.6'$.
20. $b = 112° 24.8', B = 94° 58.9'$.
21. $a = 41° 50' 20'', b = 50° 18' 11''$.
Ans. $A = 49° 19' 29''$, $B = 61° 01' 33''$, $c = 61° 35' 05''$.
22. $c = 110° 46' 20'', B = 80° 10' 30''$.
Ans. $b = 67° 06' 53''$, $a = 155° 46' 43''$, $A = 153° 58' 24''$.
23. $b = 96° 49' 59'', A = 50° 12' 04''$.
Ans. $a = 50° 00' 00''$, $B = 95° 14' 41''$, $c = 94° 23' 10''$.
24. $A = 46° 59' 42'', B = 57° 59' 17''$.
Ans. $a = 36° 27' 00''$, $b = 43° 33' 30''$, $c = 54° 20' 03''$.
25. $a = 32° 09' 17'', c = 44° 33' 17''$.
Ans. $A = 49° 20' 16''$, $b = 32° 41' 00''$, $B = 50° 19' 16''$.
26. $b = 160° 00' 00'', B = 150° 00' 00''$.
Ans. $a = 140° 55' 09''$, $A = 112° 50' 17''$, $c = 43° 09' 37''$;
$a' = 39° 04' 51''$, $A' = 67° 09' 43''$, $c' = 136° 50' 23''$.
27. $A = 60° 45' 45'', B = 57° 56' 56''$.
28. $c = 120° 23' 56'', A = 119° 34' 42''$.
29. $a = 116° 52' 45'', b = 16° 06' 06''$.
30. $A = 81° 58' 36'', a = 67° 20' 30''$.

21. Solution of Quadrantal Triangles. The polar of a quadrantal triangle is a right triangle which may be solved by the method of Art. 20 and from it the required parts of the original quadrantal triangle are obtained by means of the relations in Art. 7. Or we may apply the right triangle formulas of Art. 13 to the polar and obtain a new set of formulas for the solution of any quadrantal triangle. Thus formula (1), Art. 13, viz., $\sin A = \sin a/\sin c$, when applied to the polar triangle becomes $\sin (180° - a) = \sin (180° - A)/\sin (180° - C)$ or $\sin a = \sin A/\sin C$. Similarly we obtain each of the following formulas for the solution of quadrantal triangles, C being the angle opposite the quadrant c.

$$\sin a = \sin A/\sin C \quad (1) \qquad \tan b = \tan B/\sin A \quad (6)$$
$$\sin b = \sin B/\sin C \quad (2) \qquad \sin a = \cos b/\cos B \quad (7)$$
$$-\cos a = \tan B/\tan C \quad (3) \qquad \sin b = \cos a/\cos A \quad (8)$$
$$-\cos b = \tan A/\tan C \quad (4) \qquad -\cos C = \cos A \cos B \quad (9)$$
$$\tan a = \tan A/\sin B \quad (5) \qquad -\cos C = \cot a \cot b \quad (10)$$

EXAMPLE. Solve the quadrantal triangle in which

$$a = 97^\circ\ 24',\ A = 103^\circ\ 12',\ c = 90^\circ.$$

Solution. The polar triangle has the parts

$$A = 180^\circ - 97^\circ\ 24' = 82^\circ\ 36',\ a = 180^\circ - 103^\circ\ 12' = 76^\circ\ 48',$$
$$C = 180^\circ - 90^\circ = 90^\circ.$$

Solving this right triangle by the method of Art. 20 we find

$$B = 34^\circ\ 20', \quad b = 33^\circ\ 37', \quad c = 79^\circ\ 02';$$
$$B' = 145^\circ\ 40', \quad b' = 146^\circ\ 23', \quad c' = 100^\circ\ 58'.$$

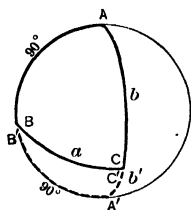



Fig. 24.

The required parts of the quadrantal triangle are, therefore,

$$b = 180^\circ - 34^\circ\ 20' = 145^\circ\ 40',$$
$$b' = 180^\circ - 145^\circ\ 40' = 34^\circ\ 20',$$
$$B = 180^\circ - 33^\circ\ 37' = 146^\circ\ 23',$$
$$B' = 180^\circ - 146^\circ\ 23' = 33^\circ\ 37',$$
$$C = 180^\circ - 79^\circ\ 02' = 100^\circ\ 58',$$
$$C' = 180^\circ - 100^\circ\ 58' = 79^\circ\ 02'.$$

Fig. 24 represents both solutions geometrically.

22. Special Formulas for Angles near 0°, 90° or 180°. An angle near 0° or 180° can not be accurately determined from its cosine, nor an angle near 90° from its sine (see Pl. Trig., Art. 21); in such cases the formulas of Art. 13 are, therefore, no longer adequate. The difficulty may be avoided by employing the following formulas:

A near 0° or 180°,	$\tan^2 \frac{1}{2} A$	$= \sin (c - b)/\sin (c + b).$	(1)
B near 0° or 180°,	$\tan^2 \frac{1}{2} B$	$= \sin (c - a)/\sin (c + a).$	(2)
a near 0° or 180°,	$\tan^2 \frac{1}{2} a$	$= \tan \frac{1}{2} (c + b) \tan \frac{1}{2} (c - b).$	(3)
b near 0° or 180°,	$\tan^2 \frac{1}{2} b$	$= \tan \frac{1}{2} (c + a) \tan \frac{1}{2} (c - a).$	(4)
c near 0° or 180°,	$\tan^2 \frac{1}{2} c$	$= -\cos (A + B)/\cos (A - B)$	(5)
A near 90°,	$\tan^2(45^\circ - \frac{1}{2} A)$	$= \tan \frac{1}{2} (c - a)/\tan \frac{1}{2} (c + a)$	(6)
		$= \tan \frac{1}{2} (B - b) \tan \frac{1}{2} (B + b).$	(7)
B near 90°,	$\tan^2 (45^\circ - \frac{1}{2} B)$	$= \tan \frac{1}{2} (c - b)/\tan \frac{1}{2} (c + b)$	(8)
		$= \tan \frac{1}{2} (A - a) \tan \frac{1}{2} (A + a).$	(9)
a near 90°,	$\tan^2 (45^\circ - \frac{1}{2} a)$	$= \sin (B - b)/\sin (B + b).$	(10)
b near 90°,	$\tan^2 (45^\circ - \frac{1}{2} b)$	$= \sin (A - a)/\sin (A + a).$	(11)
c near 90°,	$\tan^2 (45^\circ - \frac{1}{2} c)$	$= \tan \frac{1}{2} (A - a)/\tan \frac{1}{2} (A + a)$	(12)
		$= \tan \frac{1}{2} (B - b)/\tan \frac{1}{2} (B + b).$	(13)

To deduce (1) we have

$$\cos A = \tan b/\tan c, \qquad \text{(Art. 13 (3))}$$

$$\frac{1 - \cos A}{1 + \cos A} = \frac{\tan c - \tan b}{\tan c + \tan b}, \qquad \text{(Comp. and div.)}$$

$$\frac{1 - \cos A}{1 + \cos A} = \frac{1 - (1 - 2\sin^2 \frac{1}{2} A)}{1 + (2\cos^2 \frac{1}{2} A - 1)} = \tan^2 \tfrac{1}{2} A, \qquad \text{(Pl. Trig., Art. 111)}$$

and

$$\frac{\tan c - \tan b}{\tan c + \tan b} = \frac{\sin c \cos b - \cos c \sin b}{\sin c \cos b + \cos c \sin b} = \frac{\sin (c - b)}{\sin (c + b)};$$

(Pl. Trig., Art. 109)

hence $\tan^2 \frac{1}{2} A = \sin(c - b)/\sin(c + b).$

Again, to deduce (13) we proceed as follows:

$$\sin c = \sin b/\sin B, \qquad \text{(Art. 13 (2))}$$

$$\frac{1 - \sin c}{1 + \sin c} = \frac{\sin B - \sin b}{\sin B + \sin b}, \qquad \text{(Comp. and div.)}$$

$$\frac{1 - \sin c}{1 + \sin c} = \frac{1 - 2\sin \frac{1}{2} c \cos \frac{1}{2} c}{1 + 2 \sin \frac{1}{2} c \cos \frac{1}{2} c} = \frac{(\cos \frac{1}{2} c - \sin \frac{1}{2} c)^2}{(\cos \frac{1}{2} c + \sin \frac{1}{2} c)^2}$$

(Pl. Trig., Art. 111)

$$= \frac{(1 - \tan \frac{1}{2} c)^2}{(1 + \tan \frac{1}{2} c)^2} = \tan^2 (45^\circ - \tfrac{1}{2} c),$$

(Pl. Trig., Art. 110)

and

$$\frac{\sin B - \sin b}{\sin B + \sin b} = \frac{2 \cos \frac{1}{2} (B + b) \sin \frac{1}{2} (B - b)}{2 \sin \frac{1}{2} (B + b) \cos \frac{1}{2} (B - b)} = \frac{\tan \frac{1}{2} (B - b)}{\tan \frac{1}{2} (B + b)};$$

(Pl. Trig., Art. 113)

hence $\tan^2 (45^\circ - \frac{1}{2} c) = \tan \frac{1}{2} (B - b)/\tan \frac{1}{2} (B + b).$

All the other formulas given above may be deduced in a similar manner.

EXERCISE 5

1. Solve the quadrantal triangle given in Art. 21 by using formulas (8), (5), and (1) of that article.

Solve the following quadrantal triangles:

2. $C = 67^\circ\, 12'$, $b = 123^\circ\, 48'$.
   Ans. $B = 130^\circ\, 00'$, $A = 52^\circ\, 56'$, $a = 59^\circ\, 56'$.

3. $C = 141° 02.8'$, $A = 142° 05.9'$.

Ans. $B = 170° 15.0'$, $b = 164° 29.3'$, $a = 102° 10.5'$.

4. $a = 174° 12' 49''$, $b = 94° 08' 20''$.

Ans. $A = 175° 57' 10''$, $B = 135° 42' 50''$, $C = 135° 34' 07''$.

5. $a = 91° 30'$, $b = 92° 24'$.

6. $C = 136° 14.7'$, $A = 141° 45.5'$.

7. $a = 112° 56' 56''$, $C = 74° 45' 36''$.

8. In a right spherical triangle one side is $95° 52' 15''$ and the hypotenuse is $95° 44' 12''$. Find the angle opposite the given side.

Ans. $91° 15' 01''$.

9. Solve the right spherical triangle in which $a = 37° 40' 12''$, $c = 37° 40' 20''$.

Ans. $A = 89° 25' 32''$, $B = 00° 43' 32''$, $b = 00° 26' 36''$.

10. Solve the right spherical triangle in which $a = 34° 06' 13''$, $A = 34° 07' 41''$.

Ans. $b = 87° 32' 39''$, $B = 88° 37' 21''$, $c = 87° 58' 00''$.

11. Prove formulas (2), (5) and (10), Art. 22.

12. Verify formulas (3), (6) and (7), Art. 22.

23. Oblique Spherical Triangles Solved by the Method of Right Triangles. Just as every plane triangle can be solved by considering it the sum or difference of two right triangles formed by drawing a perpendicular from a vertex of the triangle to the opposite side or opposite side produced (Pl. Trig., Art. 52), so likewise every

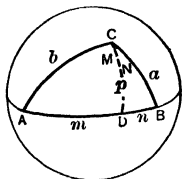



Fig. 25.

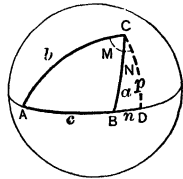



Fig. 26.

oblique spherical triangle *ABC* may be solved by considering the triangle as the sum (Fig. 25) or the difference (Fig. 26) of the two right triangles *ACD* and *BCD* formed by the perpendicular arc of a great circle drawn from one of the vertices to the opposite side or opposite side produced.

We shall denote by m and n the segments AD and DB into which the perpendicular $p = CD$ divides the opposite side c, and by M and N the angles ACD and DCB into which the angle C is divided by the same perpendicular. We then have

$c = m + n$, $C = M + N$ (Fig. 25); $c = m - n$, $C = M - N$ (Fig. 26).

The method of solving oblique spherical triangles by dividing them into right triangles, while exceedingly simple in principle, is not the most convenient method nor the method commonly employed in actual computation. Better methods will be developed in the next chapter and the student is expected to familiarize himself with the methods there presented rather than to depend on the method of the present article.

Case III. Given two sides and the included angle, b, c, A.

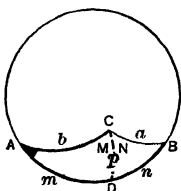



Fig. 27.

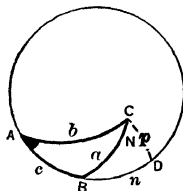



Fig. 28.

Solution. 1. In triangle ACD find p, M and m.
2. $n = c - m$ (Fig. 27), or $n = m - c$ (Fig. 28).
3. In triangle BCD find N, a and B.
4. $C = M + N$ (Fig. 27), or $C = M - N$ (Fig. 28).
5. Check. Repeat the solution drawing the perpendicular from B to the side AC.

Case IV. Given two angles and the included side, B, C, a.

Solution. Solve the polar triangle by Case III and then compute the unknown parts of the original triangle.

Case V. Given two sides and the angle opposite one of them, a, b, A.

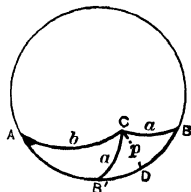



Fig. 29.

Solution. In this case there are two solutions, provided that a is intermediate in value between p and both b and $180° - b$ (Art. 11).

1. In triangle ACD find p, $M = ACD$, and $m = AD$.

2. In triangle BCD find $N = BCD$, B, and $n = DB$, $AB'C = 180° - B$.

3. $ACB = M + N, ACB' = M - N, AB = m + n, AB' = m - n$.

4. Check. Assume b, c, A as the given parts and find the other parts by Case III.

Case VI. Given two angles and the side opposite one of them, A, B, a.

Solution. Solve the polar triangle by Case V and from it find the unknown parts of the original triangle. As there may be two solutions in Case V so Case VI may have two solutions.

Case I. Given the three sides, a, b, c.

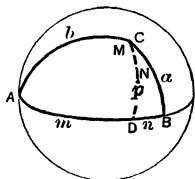



Fig. 30.

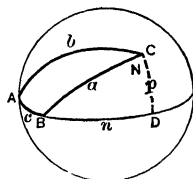



Fig. 31.

Solution. In the triangle ACD we have by Napier's rule

$$\sin \bar{b} = \cos p \cos m, \text{ or } \cos p = \cos b / \cos m.$$

Similarly we have in the triangle BCD

$$\sin \bar{a} = \cos p \cos n, \text{ or } \cos p = \cos a / \cos n.$$

Hence

$$\frac{\cos a}{\cos b} = \frac{\cos m}{\cos n}, \text{ from which } \frac{\cos a - \cos b}{\cos a + \cos b} = \frac{\cos m - \cos n}{\cos m + \cos m}.$$

Now

$$\frac{\cos a - \cos b}{\cos a + \cos b} = \frac{-2 \sin\frac{1}{2}(a+b) \sin\frac{1}{2}(a-b)}{2 \cos\frac{1}{2}(a+b) \cos\frac{1}{2}(a-b)} = -\tan\tfrac{1}{2}(a+b) \tan\tfrac{1}{2}(a-b),$$

so that

$$\tan\tfrac{1}{2}(a+b) \tan\tfrac{1}{2}(a-b) = \tan\tfrac{1}{2}(m+n) \tan\tfrac{1}{2}(m-n),$$

from which

$$\tan\tfrac{1}{2}(m-n) = \tan\tfrac{1}{2}(a+b) \tan\tfrac{1}{2}(a-b) \cot\tfrac{1}{2} c,$$

if $$m + n = c \text{ (Fig. 30)},$$

or $$\tan \tfrac{1}{2}(m + n) = \tan \tfrac{1}{2}(a + b) \tan \tfrac{1}{2}(a - b) \cot \tfrac{1}{2} c,$$

if $$m - n = c \text{ (Fig. 31)}.$$

We have, therefore, the following steps:

1. Find $\frac{1}{2}(m - n)$ (Fig. 30), or $\frac{1}{2}(m + n)$ (Fig. 31), from the relation

$$\tan \tfrac{1}{2}(m - n) = \tan \tfrac{1}{2}(a + b) \tan \tfrac{1}{2}(a - b) \cot \tfrac{1}{2} c,$$
$$\tan \tfrac{1}{2}(m + n) = \tan \tfrac{1}{2}(a + b) \tan \tfrac{1}{2}(a - b) \cot \tfrac{1}{2} c.$$

2. $m = \frac{1}{2}(m + n) + \frac{1}{2}(m - n)$, $n = \frac{1}{2}(m + n) - \frac{1}{2}(m - n)$.

3. In triangle ACD find A and M.

4. In triangle BCD find B and N.

5. $C = M + N$ (Fig. 30), or $C = M - N$ (Fig. 31).

6. Check. Repeat the solution drawing the perpendicular from B on AC or from A on BC.

Case II. Given the three angles, A, B, C.

Solution. Solve the polar triangle by Case I, and from it compute the unknown parts of the original triangle.

Exercise 6

1. Show how Case IV may be solved by means of right triangles without using the polar triangle, and outline the steps of the solution.

2. Prove *Bowditch's Rules for Oblique Spherical Triangles* which may be stated as follows: If a spherical triangle is divided into two right triangles by a perpendicular let fall from one of the vertices to the opposite side, and if in the two right triangles the middle parts are so chosen that the perpendicular is an adjacent part in each triangle, then

The sines of the middle parts in the two triangles are proportional to the tangents of the adjacent parts;

but if the perpendicular is an opposite part in each triangle, then

The sines of the middle parts are proportional to the cosines of the opposite parts.

As in the case of Napier's rules, the parts referred to in these rules are the circular parts of the two triangles. By the use of Bowditch's rules the solution of oblique spherical triangles by means of right triangles may be somewhat shortened.

Solve the following triangles by means of right triangles:

3. Given $b = 88^\circ 24'$, $c = 56^\circ 48'$, $A = 128^\circ 16'$;
find $B = 65^\circ 13'$, $C = 49^\circ 28'$, $a = 120^\circ 11'$.

4. Given $a = 103^\circ 44'$, $b = 65^\circ 12'$, $C = 97^\circ 34'$.

5. Given $a = 148^\circ 34.4'$, $b = 142^\circ 11.6'$, $A = 153^\circ 17.6'$;
find $c = 62^\circ 08.6'$, $B = 148^\circ 06.3'$, $C = 130^\circ 21.2'$,
$c' = 7^\circ 18.4'$, $B' = 31^\circ 53.7'$, $C' = 6^\circ 17.6'$.

6. Given $A = 110^\circ$, $B = 62^\circ$, $a = 49^\circ$.

7. Given $A = 80^\circ 20.2'$, $B = 73^\circ 46.7'$, $C = 54^\circ 08.5'$;
find $a = 64^\circ 47.2'$, $b = 61^\circ 47.3'$, $c = 48^\circ 03.4'$.

8. Given $a = 31^\circ 11' 07''$, $b = 32^\circ 19' 18''$, $c = 33^\circ 15' 21''$;
find $A = 59^\circ 29' 42''$, $B = 62^\circ 49' 42''$, $C = 65^\circ 50' 48''$.

9. Given $a = 87^\circ 45' 24''$, $b = 96^\circ 12' 15''$, $c = 100^\circ 08' 56''$.

10. Given $A = 87^\circ 45' 24''$, $B = 96^\circ 12' 15''$, $C = 100^\circ 08' 56''$.

CHAPTER III

PROPERTIES OF OBLIQUE SPHERICAL TRIANGLES

WE shall now develop a number of formulas involving the parts of any spherical triangle, from which, if any three parts of the triangle are given, the remaining parts may be derived by computation without first dividing the triangle into right triangles as was done in the last article. Then, in order to facilitate the work of computation, we shall transform these formulas so as to adapt them to the use of logarithms. The actual application of the formulas to the solution of triangles we shall reserve for a separate chapter.

24. The Law of Sines. (*a*) *First Proof.* Let ABC be any spherical triangle, p the perpendicular from one of the vertices C of the triangle to the opposite side AB (Fig. 32) or AB produced (Fig. 33).

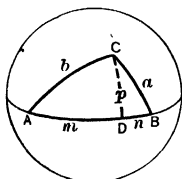



Fig. 32

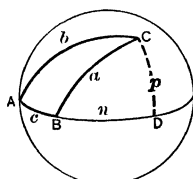



Fig. 33.

By Napier's rules, or the formulas of Art. 13, we have

from triangle ACD $\quad \sin p = \sin b \sin A,$

and from triangle BCD $\quad \sin p = \sin a \sin B \qquad (B \text{ acute}),$

or $\quad \sin p = \sin a \sin (180^\circ - B)$

$\qquad = \sin a \sin B \qquad (B \text{ obtuse}).$

Hence, whether the perpendicular falls within the triangle or without, we have

$$\sin p = \sin a \sin A = \sin a \sin B.$$

Advancing letters, $\quad \sin c \sin B = \sin b \sin C,$

$$\sin a \sin C = \sin c \sin A$$

These equations may also be written in the form

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}, \tag{1}$$

or in words, *The sines of the sides of a spherical triangle are proportional to the sines of the opposite angles.*

(*b*) *Second Proof.* Let ABC (Fig. 34) be a spherical triangle and O the center of the sphere on which the triangle lies. Draw the radii OA, OB, OC. From C draw CD perpendicular to the plane of AOB and through CD draw planes CDE and CDF perpendicular to OA and OB respectively. Then each of the triangles, OEC, CDE, CDF, OFC, is right-angled, the middle letter being in each case at the right angle. Also since CF and DF are perpendicular to OB, angle CFD is equal to the angle B, and similarly angle CED is equal to the angle A.

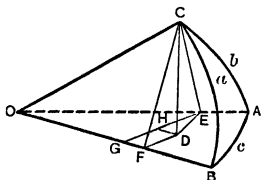



Fig. 34.

Now $\quad CD = CE \sin CED = CE \sin A,$

and $\quad CD = CF \sin CFD = CF \sin B,$

$$CE = OC \sin COE = OC \sin b,$$
$$CF = OC \sin COF = OC \sin a.$$

Therefore, substituting in the first two equations for CE and CF their values from the last two, we have

$$OC \sin b \sin A = OC \sin a \sin B,$$

from which

$$\sin a/\sin A = \sin b/\sin B.$$

25. The Law of Cosines. (*a*) *First Proof.* In Figs. 32 and 33 let us denote AD and DB by m and n respectively. By applying Napier's rules, or the formulas of Art. 13, we find

from triangle BCD $\qquad \cos a = \cos p \cos n,$

and from triangle ACD $\qquad \cos b = \cos p \cos m.$

Now $\qquad n = c - m$ (B acute), or $n = m - c$ (B obtuse),

and since $\cos (c - m) = \cos (m - c)$, we have in either case on eliminating $\cos p$ and putting for n its value

$$\begin{aligned}\cos a &= \cos b \cos (c - m)/\cos m \\ &= \cos b \frac{\cos c \cos m + \sin c \sin m}{\cos m} \\ &= \cos b \cos c + \cos b \sin c \tan m.\end{aligned}$$

But by Napier's rules $\tan m = \tan b \cos A$, hence substituting this value in the last equation and remembering that $\cos b \tan b = \sin b$, we have

Advancing letters,
$$\left.\begin{aligned}\cos a &= \cos b \cos c + \sin b \sin c \cos A.\\ \cos b &= \cos c \cos a + \sin c \sin a \cos B,\\ \cos c &= \cos a \cos b + \sin a \sin b \cos C.\end{aligned}\right\} \quad (2)$$

These formulas embody the *Law of Cosines: The cosine of any side of a spherical triangle is equal to the product of the cosines of the other two sides plus the continued product of the sines of these two sides and the cosine of the included angle.*

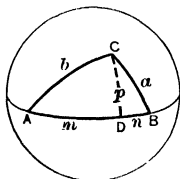



Fig. 32.

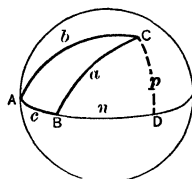



Fig 33.

(*b*) *Second Proof.* In Fig. 34 draw EG parallel to DF and DH perpendicular to EG, then angle DEH equals angle AOB or c, and we have

$$\frac{HD}{OC} = \frac{HD}{DE} \cdot \frac{DE}{CE} \cdot \frac{CE}{OC} = \sin c \cos A \sin b,$$

$$\frac{HD}{OC} = \frac{OF}{OC} - \frac{OG}{OC} = \frac{OF}{OC} - \frac{OG}{OE} \cdot \frac{OE}{OC} = \cos a - \cos c \cos b.$$

Equating these two values of HD/OC and solving for $\cos a$ we find

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

26. Relation Between Two Angles and Three Sides.

The second of the equations (2) may be written

$$\cos c \cos a + \sin c \sin a \cos B = \cos b,$$

and the first multiplied by $\cos c$ gives

$$\cos c \cos a = \cos b \cos^2 c + \sin b \sin c \cos c \cos A.$$

Subtracting the second of these equations from the first gives

$$\sin c \sin a \cos B = \cos b\,(1 - \cos^2 c) - \sin b \sin c \cos c \cos A.$$

Now $1 - \cos^2 c = \sin^2 c$, hence we may divide the equation by $\sin c$, and obtain

$$\left.\begin{aligned}
\sin a \cos B &= \cos b \sin c - \sin b \cos c \cos A.\\
\text{Similarly,}\quad \sin a \cos C &= \cos c \sin b - \sin c \cos b \cos A.\\
\sin b \cos C &= \cos c \sin a - \sin c \cos a \cos B,\\
\sin b \cos A &= \cos a \sin c - \sin a \cos c \cos B,\\
\sin c \cos A &= \cos a \sin b - \sin a \cos b \cos C,\\
\sin c \cos B &= \cos b \sin a - \sin b \cos a \cos C.
\end{aligned}\right\} \quad (3)$$

27. Third Proof of the Fundamental Formulas. The three equations (1) Art. 24, (2) Art. 25, and (3) Art. 26, may be derived simultaneously by the method of analytical geometry.* Let ABC be any spherical triangle. Take O, the center of the sphere, for the origin of a system of rectangular coordinates, the plane of BOA for the xy-plane, OB for the direction of the x-axis, and the positive z-axis on the same side of the plane BOA as the vertex C. Join O and C. From C drop the perpendicular CR on BOY, and through CR pass a plane perpendicular to OB cutting OB in S. Then the triangles CRS and CSO have right angles at R and S respectively, and angle RSC equals angle B (why?). Denoting the coordinates of C by x, y, z and the distance OC by r, we have

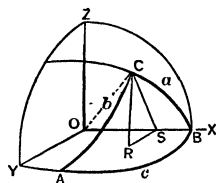



Fig. 35.

$$\begin{aligned}
OS &= OC \cos COS, & \text{or } x &= r\cos a,\\
RS &= SC \cos RSC = OC \sin COS \cos RSC, & \text{or } y &= r \sin a \cos B,\\
RC &= SC \sin RSC = OC \sin COS \sin RSC, & \text{or } z &= r \sin a \sin B.
\end{aligned}$$

If OA had been taken for the x-axis, the z-axis remaining unchanged, A and a will change places with B and b respectively, and the y coordinates will have opposite signs, so that the new coördinates x', y', z' of C will be

$$x' = r\cos b, \quad y' = - r \sin b \cos A, \quad z' = r \sin b \sin A.$$

But these are the transformed coördinates of a system having the same z-axis while the x- and y-axes are each turned through an angle c,

* The student without some knowledge of analytical geometry must content himself with the proofs given in the preceding articles and those suggested in the exercises which follow.

hence the coördinates x, y, z and x', y', z' are related by the transformation formulas,

$$z = z', \quad x = x' \cos c - y' \sin c, \quad y = x' \sin c + y' \cos c.$$

Substituting in these three formulas the values of x, y, z, x', y', z' in terms of r and the parts of the triangle, we have, after dividing out r,

$$\mathbf{\sin a \sin B = \sin b \sin A,} \tag{1}$$
$$\mathbf{\cos a = \cos b \cos c + \sin b \sin c \cos A,} \tag{2}$$
$$\mathbf{\sin a \cos B = \cos b \sin c - \sin b \cos c \cos A.} \tag{3}$$

28. Fundamental Relations for the Polar Triangle. If we apply the formulas (1), (2), (3) to the polar triangle, by putting $a = 180° - A'$, $A = 180° - a'$, etc. (Art. 7), and then drop the accents, we find that (1) remains unchanged, while (2) and (3) give rise to the new sets of formulas:

$$\left.\begin{aligned} \cos A &= -\cos B \cos C + \sin B \sin C \cos a, \\ \cos B &= -\cos C \cos A + \sin C \sin A \cos b, \\ \cos C &= -\cos A \cos B + \sin A \sin B \cos c, \end{aligned}\right\} \tag{4}$$

and

$$\left.\begin{aligned} \sin A \cos b &= \cos B \sin C + \sin B \cos C \cos a, \\ \sin A \cos c &= \cos C \sin B + \sin C \cos B \cos a, \\ \sin B \cos c &= \cos C \sin A + \sin C \cos A \cos b, \\ \sin B \cos a &= \cos A \sin C + \sin A \cos C \cos b, \\ \sin C \cos a &= \cos A \sin B + \sin A \cos B \cos c, \\ \sin C \cos b &= \cos B \sin A + \sin B \cos A \cos c. \end{aligned}\right\} \tag{5}$$

29. Arithmetic Solution of Spherical Triangles. The fundamental relations (1), (2), (3) enable us to solve every case of oblique spherical triangles.

Case I. *Given the three sides, a, b, c.*

1. The angle A may be found by the law of cosines.
2. The angles B and C may then be found by the law of sines.

Case III. *Given two sides and the included angle, a, b, C.*

1. The third side may be found by the law of cosines.
2. The angles A and B may then be found by the law of sines.

Case V. *Given two sides and the angle opposite one of them, a, b, A.*

1. The angle B may be found by the law of sines.
2. The third side might be found by the law of cosines but since the law of cosines involves both $\sin c$ and $\cos c$ the formula solved for

either $\sin c$ or $\cos c$ would involve radical expressions. These may be avoided by using the formula

$$\cos c = \frac{\cos a \cos b - \sin a \sin b \cos A \cos B}{1 - \sin^2 b \sin^2 A},$$

which is obtained by eliminating $\sin c$ from the formulas (2) and (3) of Art. 27.

3. The angle C may now be found by the law of sines.

Cases II, IV, VI. These may be referred to Cases I, III, V, respectively, by making use of the polar triangle, or we may apply formulas (1), (4), (5).

While the fundamental relations (1), (2), (3) make it possible to solve each of the six cases of triangles, it is clear that (2) and (3) are not adapted to logarithmic computation. Therefore, in order to facilitate computation, it is desirable to obtain other formulas which enable us to use logarithms. Such formulas will be developed in the following articles.

EXERCISE 7

1. If a', b', c' denote the sides of the polar triangle, show that

$$\sin a : \sin b : \sin c = \sin a' : \sin b' : \sin c'.$$

2. If m is the arc joining the vertex C of a spherical triangle to the middle point of the opposite side, show that

$$\cos a + \cos b = 2 \cos m \cos \tfrac{1}{2} c.$$

3. If the bisector of the angle C meets the opposite side in D, show that

$$\sin a : \sin b = \sin BD : \sin AD.$$

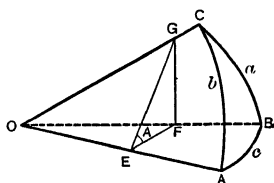



Fig. 36.

4. State in words the laws expressed by formulas (4) and (5), Art. 28.

5. In Fig. 36 let EGF be the triangle in which a plane drawn perpendicular to an edge OA intersects the trihedral angle. Then

$$GF^2 = OF^2 + OG^2 - 2\,OF \cdot OG \cdot \cos a.$$
$$GF^2 = EF^2 + EG^2 - 2\,EF \cdot EG \cdot \cos A.$$

Subtracting and observing that $OF^2 - EF^2 = OE^2$, $OG^2 - EG^2 = OE^2$, we find

$$2\,OF \cdot OG \cdot \cos a = 2\,OE^2 + 2\,EF \cdot EG \cdot \cos A,$$

which, on dividing by $2\ OF \cdot OG$, leads to

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

This constitutes a *fourth proof of the law of cosines.*

6. From the law of cosines

$$\cos A = (\cos a - \cos b \cos c)/\sin b \sin c,$$

show that

$$\frac{\sin^2 A}{\sin^2 a} = \frac{1 - \cos^2 A}{\sin^2 a} = \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 a \sin^2 b \sin^2 c}.$$

The expression on the right is symmetrical in a, b, and c, hence

$$\frac{\sin^2 A}{\sin^2 a} = \frac{\sin^2 B}{\sin^2 b} = \frac{\sin^2 C}{\sin^2 c}, \text{ from which } \frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

This constitutes a *fourth proof of the law of sines.*

7. Prove the relation

$$\cot a \sin b = \cot A \sin C + \cos C \cos b.$$

Suggestion. Multiply the third of the equations (2), Art. 25, by $\cos b$, substitute in the first equation and divide by $\sin b \sin c$.

8. By interchanging and advancing letters write down five other equations like that in Problem 7.

9. Apply the relations of Problems 7 and 8 to the polar triangle. Do the resulting equations express new relations?

10. Given $b = 135°$, $c = 45°$, $A = 60°$; find the remaining parts to the nearest degree.

Ans. $a = 104°$, $B = 141°$, $C = 39°$.

11. Given $a = 120°$, $b = 60°$, $A = 135°$; find the remaining parts to the nearest minute.

Ans. $B = 45°\ 00'$, $c = 78°\ 28'$, $C = 53°\ 08'$.

12. Given $a = 135°$, $b = 135°$, $c = 45°$; find A, B, C, to the nearest minute. *Ans.* $A = B = 114°\ 28'$, $C = 65°\ 32'$.

30. Functions of Half the Angles in Terms of the Sides.

From the law of cosines

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} = 1 - 2 \sin^2 \tfrac{1}{2} A, \quad \text{(Pl. Trig., Art. 111)}$$

therefore

$$2 \sin^2 \tfrac{1}{2} A = 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c} = \frac{\cos (b - c) - \cos a}{\sin b \sin c}.$$

Now

$$\cos(b-c)-\cos a = 2\sin\tfrac{1}{2}(a+b-c)\sin\tfrac{1}{2}(a-b+c) \quad \text{(Pl. Trig., Art. 113)}$$
$$= 2\sin(s-c)\sin(s-b), \text{ where } s=\tfrac{1}{2}(a+b+c),$$

therefore

$$2\sin^2\tfrac{1}{2}A = \frac{2\sin(s-b)\sin(s-c)}{\sin b \sin c},$$

or

$$\left.\begin{aligned} \sin\tfrac{1}{2}A &= \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}}. \\ \text{Similarly,}\quad \sin\tfrac{1}{2}B &= \sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin c\sin a}}, \\ \sin\tfrac{1}{2}C &= \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a\sin b}}, \end{aligned}\right\} \quad (6)$$

$$s=\tfrac{1}{2}(a+b+c).$$

Corresponding formulas for the cosines of half the angles may be obtained by applying the formulas (6) to the co-lunar triangles. Thus by applying the first formula to the co-lunar triangle $AB'C$ whose parts are (Art. 5) $180° - A$, B, $180° - C$, $180° - a$, b, $180° - c$,

$$\left.\begin{aligned} \text{we obtain}\quad \cos\tfrac{1}{2}A &= \sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}}. \\ \text{Similarly,}\quad \cos\tfrac{1}{2}B &= \sqrt{\frac{\sin s\sin(s-b)}{\sin c\sin a}}, \\ \cos\tfrac{1}{2}C &= \sqrt{\frac{\sin s\sin(s-c)}{\sin a\sin b}}. \end{aligned}\right\} \quad (7)$$

To find $\tan\frac{1}{2}A$ we divide $\sin\frac{1}{2}A$ by $\cos\frac{1}{2}A$ and obtain

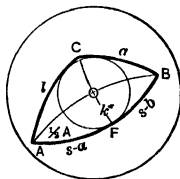
$$\left.\begin{aligned} \tan\tfrac{1}{2}A &= \frac{\tan k}{\sin(s-a)}, \\ \tan\tfrac{1}{2}B &= \frac{\tan k}{\sin(s-b)}, \\ \tan\tfrac{1}{2}C &= \frac{\tan k}{\sin(s-c)}, \\ \text{where}\quad \tan k &= \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}. \end{aligned}\right\} \quad (8)$$

k is the arcual radius of the small circle inscribed in the triangle ABC, for if O (Fig. 37) represents the intersection of the arcs bisect-

ing the angles of the triangle, OF, the arc drawn from O perpendicular to one of the sides as AB, will be the arcual radius of the inscribed circle. It follows, just as in the case of plane triangles (Pl. Trig., Art. 68), that $AF = s - a$, hence denoting OF by k and applying Napier's rules to the right triangle AOF, we have

Fig. 37.

$$\sin (s - a) = \cot \tfrac{1}{2} A \tan k,$$

or
$$\tan \tfrac{1}{2} A = \tan k/\sin (s - a).$$

31. Functions of Half the Sides in Terms of the Angles. If we apply the formulas (6) and (7), Art. 30, to the polar triangle (Art. 7), by putting $A = 180° - a'$, $a = 180° - A'$, $B = 180° - b'$, etc., dropping the accents in the final results, we obtain

$$\left.\begin{aligned} \sin \tfrac{1}{2} a &= \sqrt{\frac{-\cos S \cos (S - A)}{\sin B \sin C}} \\ \sin \tfrac{1}{2} b &= \sqrt{\frac{-\cos S \cos (S - B)}{\sin C \sin A}} \\ \sin \tfrac{1}{2} c &= \sqrt{\frac{-\cos S \cos (S - C)}{\sin A \sin B}} \end{aligned}\right\} \quad (9)$$

$$\left.\begin{aligned} \cos \tfrac{1}{2} a &= \sqrt{\frac{\cos (S - B) \cos (S - C)}{\sin B \sin C}} \\ \cos \tfrac{1}{2} b &= \sqrt{\frac{\cos (S - C) \cos (S - A)}{\sin C \sin A}} \\ \cos \tfrac{1}{2} c &= \sqrt{\frac{\cos (S - A) \cos (S - B)}{\sin A \sin B}} \end{aligned}\right\} \quad (10)$$

$$S = \tfrac{1}{2}(A + B + C).$$

From (9) and (10) we find

$$\left.\begin{aligned} \tan \tfrac{1}{2} a &= \tan K \cos (S - A), \\ \tan \tfrac{1}{2} b &= \tan K \cos (S - B), \\ \tan \tfrac{1}{2} c &= \tan K \cos (S - C), \end{aligned}\right.$$

where

$$\left.\tan K = \sqrt{\frac{-\cos S}{\cos (S - A) \cos (S - B) \cos (S - C)}}.\right\} \quad (11)$$

K is the arcual radius of the small circle circumscribed about the triangle ABC, for if O (Fig. 38) is the center of this circle, OA, OB, OC the arcs joining the center to the vertices of the triangle, OF the perpendicular arc from O to one of the sides as BC, then $OA = OB = OC = K$, the triangles AOB, BOC, COA are isosceles, and $BF = FC = \frac{1}{2}a$. Furthermore

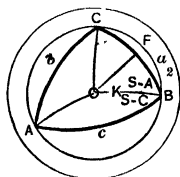



Fig. 38.

$$A = BAO + OAC = ABO + ACO = (B - OBF) + (C - OCF) = B + C - 2\,OBF,$$

hence $$OBF = \tfrac{1}{2}(B + C - A) = S - A,$$

where $$S = \tfrac{1}{2}(A + B + C).$$

If now we apply Napier's rules to the right triangle BOF, we find

$$\cos OBF = \cot BO \tan BF$$

or $$\cos (S - A) = \cot K \tan \tfrac{1}{2} a,$$

from which $$\tan \tfrac{1}{2} a = \tan K \cos (S - A).$$

EXERCISE 8

1. Prove the formula for $\sin \frac{1}{2} C$ (Art. 30) directly by using the relation

$$\cos C = 1 - 2 \sin^2 \tfrac{1}{2} C = \frac{\cos c - \cos a \cos b}{\sin a \sin b}.$$

2. Prove the formula for $\cos \frac{1}{2} A$ (Art. 30) directly by using the relation

$$\cos A = 2 \cos^2 \tfrac{1}{2} A - 1 = \frac{\cos a - \cos b \cos c}{\sin b \sin c},$$

and following the method used in deriving the formula for $\sin \frac{1}{2} A$.

3. Prove the formula for $\sin \frac{1}{2} a$ (Art. 31) directly by using the relation

$$\cos a = 1 - 2 \sin^2 \tfrac{1}{2} a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}.$$

4. Prove the formula for $\cos \frac{1}{2} a$ (Art. 31) directly by using the relation

$$\cos a = 2 \cos^2 \tfrac{1}{2} a - 1 = \frac{\cos A + \cos B \cos C}{\sin B \sin C}.$$

5. Derive the formula for $\tan \frac{1}{2} a$ (Art. 31) by applying the formula for $\tan \frac{1}{2} A$ (Art. 30) to the polar triangle.

6. Derive the formula for $\cos \frac{1}{2} A$ by applying the formula for $\sin \frac{1}{2} A$ to the co-lunar triangle ABC'.

7. Apply the formula for $\sin \frac{1}{2} A$ to the co-lunar triangle $A'BC$. Does the resulting formula express a new relation?

8. The escribed circles of the triangle ABC are the small circles inscribed in the co-lunar triangles $A'BC$, $AB'C$, ABC'. By applying the formula for $\tan k$ (Art. 30) to these triangles, show that the arcual radii, k_a, k_b, k_c of the escribed circles are given by the formulas

$$\tan k_a = \sqrt{\frac{\sin s \sin (s-b) \sin (s-c)}{\sin (s-a)}} = \sin s \tan \tfrac{1}{2} A,$$

$$\tan k_b = \sin s \tan \tfrac{1}{2} B, \qquad \tan k_c = \sin s \tan \tfrac{1}{2} C.$$

9. By applying the formula for $\tan K$ (Art. 31) to the co-lunar triangle $A'BC$, show that the arcual radius of the circle circumscribing this triangle is given by the formula

$$\tan K_A = \sqrt{\frac{\cos (S-A)}{-\cos S \cos (S-B) \cos (S-C)}} = -\tan \tfrac{1}{2} a / \cos S,$$

hence also $\tan K_B = -\tan \frac{1}{2} b / \cos S$, $\tan K_C = -\tan \frac{1}{2} c / \cos S$.

10. Show that

$$2 \tan K = \cot k_a + \cot k_b + \cot k_c - \cot k,$$

and

$$2 \cot k = \tan K_A + \tan K_B + \tan K_C - \tan K.$$

32. Delambre's (or Gauss's) Proportions. By Pl. Trig., Art. 106, we have

$$\sin \tfrac{1}{2}(A+B) = \sin \tfrac{1}{2} A \cos \tfrac{1}{2} B + \cos \tfrac{1}{2} A \sin \tfrac{1}{2} B.$$

Substituting for $\sin \frac{1}{2} A$, $\cos \frac{1}{2} B$, $\cos \frac{1}{2} A$, $\sin \frac{1}{2} B$, their values from (6) and (7), Art. 30,

$$\begin{aligned}
\sin \tfrac{1}{2}(A+B) &= \sqrt{\frac{\sin (s-b) \sin (s-c) \sin s \sin (s-b)}{\sin a \sin b \sin^2 c}} \\
&\quad + \sqrt{\frac{\sin s \sin (s-a) \sin (s-c) \sin (s-a)}{\sin a \sin b \sin^2 c}} \\
&= \sqrt{\frac{\sin s \sin (s-c)}{\sin a \sin b}} \cdot \frac{\sin (s-b) + \sin (s-a)}{\sin c} \\
&= \cos \tfrac{1}{2} C \frac{\sin (s-b) + \sin (s-a)}{\sin c}.
\end{aligned}$$

Also by Pl. Trig., Art. 111, 113,

$$\sin(s-b)+\sin(s-a) = 2\sin\tfrac{1}{2}(s-b+s-a)\cos\tfrac{1}{2}(s-b-s+a)$$
$$= 2\sin\tfrac{1}{2}c\cos\tfrac{1}{2}(a-b),$$

and

$$\sin c = 2\sin\tfrac{1}{2}c\cos\tfrac{1}{2}c,$$

hence

$$\frac{\sin(s-b)+\sin(s-a)}{\sin c} = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c},$$

and

$$\sin\tfrac{1}{2}(A+B) = \frac{\cos\frac{1}{2}(a-b)\cos\frac{1}{2}C}{\cos\frac{1}{2}c}.$$

Similarly, we obtain corresponding formulas for $\cos\frac{1}{2}(A+B)$, $\sin\frac{1}{2}(A-B)$ and $\cos\frac{1}{2}(A-B)$. The four formulas, of which the third and fourth may also be obtained by applying the first and second to either one of the co-lunar triangles $A'BC$ or $AB'C$, may be written

$$\left.\begin{aligned}\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}c &= \cos\tfrac{1}{2}(a-b)\cos\tfrac{1}{2}C,\\ \cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}c &= \cos\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}C,\\ \sin\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}c &= \sin\tfrac{1}{2}(a-b)\cos\tfrac{1}{2}C,\\ \cos\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}c &= \sin\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}C.\end{aligned}\right\} \quad (12)$$

These formulas are known as *Delambre's* or *Gauss's* proportions or equations.

33. Napier's Proportions. If of the equations (12) we divide the first by the second, then the third by the fourth, then the fourth by the second, and finally the third by the first, we obtain the following four new formulas which are known as *Napier's* proportions or analogies.

$$\left.\begin{aligned}\tan\tfrac{1}{2}(A+B) &= \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C,\\ \tan\tfrac{1}{2}(A-B) &= \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C,\\ \tan\tfrac{1}{2}(a+b) &= \frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c,\\ \tan\tfrac{1}{2}(a-b) &= \frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c.\end{aligned}\right\} \quad (13)$$

The second of these formulas may also be obtained by applying the first to either of the co-lunar triangles $A'BC$ or $AB'C$, and the third and fourth by applying the first and second to the polar triangle.

If we divide the first of the equations (13) by the second, or the third by the fourth, we obtain the *law of tangents*

$$\frac{\tan \frac{1}{2}(a+b)}{\tan \frac{1}{2}(a-b)} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}.$$

34. Formulas for the Area of a Spherical Triangle. It is shown in Solid Geometry that the area of a spherical triangle is given by the formula

$$T = \frac{\pi R^2 E^\circ}{180^\circ}, \tag{14}$$

where R is the radius of the sphere, and E° the *Spherical Excess* expressed in degrees, that is $E^\circ = A + B + C - 180^\circ$.

If E is the spherical excess expressed in radians, $E = E^\circ \cdot \pi/180$, and

(14) becomes $$T = R^2 E. \tag{15}$$

For a unit sphere ($R = 1$) $$T = E, \tag{16}$$

hence we have

Theorem I. The area of a spherical triangle on a unit sphere is equal to the spherical excess expressed in radians.

Theorem II. The area of a spherical triangle on any sphere is equal to the area of the corresponding triangle on a unit sphere multiplied by the square of the radius.

The problem of finding various expressions for the area of a spherical triangle resolves itself, therefore, into the problem of finding various expressions for the spherical excess E.

(*a*) *In terms of the angles, A, B, C.*

$$E = 2S - \pi, \quad \text{where} \quad S = \tfrac{1}{2}(A + B + C). \tag{17}$$

(*b*) *In terms of the sides, a, b, c.*

We have

$$\sin \tfrac{1}{2} E = \sin (S - \tfrac{1}{2}\pi) = \sin [\tfrac{1}{2}(A+B) + \tfrac{1}{2}(C-\pi)]$$
$$= \sin \tfrac{1}{2}(A+B) \sin \tfrac{1}{2} C - \cos \tfrac{1}{2}(A+B) \cos \tfrac{1}{2} C.$$

Substituting for $\sin \frac{1}{2}(A+B)$ and $\cos \frac{1}{2}(A+B)$ their values from (12), we have

$$\sin \tfrac{1}{2} E = \frac{\sin \frac{1}{2} C \cos \frac{1}{2} C}{\cos \frac{1}{2} c} [\cos \tfrac{1}{2}(a-b) - \cos \tfrac{1}{2}(a+b)]$$
$$= \frac{\sin \frac{1}{2} C \cos \frac{1}{2} C}{\cos \frac{1}{2} c} (2 \sin \tfrac{1}{2} a \sin \tfrac{1}{2} b).$$

Finally by putting for $\sin\frac{1}{2}C$ and $\cos\frac{1}{2}C$ their values from (6) and (7) we arrive at

Cagnoli's Formula,

$$\sin\tfrac{1}{2}E = \frac{n}{2\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c}, \tag{18}$$

where $n = \sqrt{\sin s\sin(s-a)\sin(s-b)\sin(s-c)}$.

Or we may proceed as follows: $\frac{1}{2}(C-E) = \frac{1}{2}\pi - \frac{1}{2}(A+B)$,

and therefore, $\sin\frac{1}{2}(C-E) = \cos\frac{1}{2}(A+B)$.

This value substituted in the second of the equations (12) gives

$$\sin\tfrac{1}{2}(C-E) : \sin\tfrac{1}{2}C = \cos\tfrac{1}{2}(a+b) : \cos\tfrac{1}{2}c.$$

From this proportion we have by division and composition

$$\frac{\sin\frac{1}{2}C - \sin\frac{1}{2}(C-E)}{\sin\frac{1}{2}C + \sin\frac{1}{2}(C-E)} = \frac{\cos\frac{1}{2}c - \cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}c + \cos\frac{1}{2}(a+b)}.$$

On reducing each member of this equation by means of the relations of Art. 113 (Pl. Trig.), we obtain

$$\tan\tfrac{1}{4}E\cot\tfrac{1}{4}(2C-E) = \tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-c);$$

In like manner, by substituting $\cos\frac{1}{2}(C-E) = \sin\frac{1}{2}(A+B)$ in the first of the equations (12), we find

$$\tan\tfrac{1}{4}E\tan\tfrac{1}{4}(2C-E) = \tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b);$$

hence on multiplying these two equations and extracting the square-root we obtain

Lhuilier's Formula,

$$\tan\tfrac{1}{4}E = \sqrt{\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c)}. \tag{19}$$

(*c*) *In terms of two sides and the included angle, a, b, C.*

$$\tan\tfrac{1}{2}E = \frac{\sin(S-\frac{1}{2}\pi)}{\cos(S-\frac{1}{2}\pi)} = \frac{-\cos\frac{1}{2}(A+B+C)}{\sin\frac{1}{2}(A+B+C)}$$

$$= \frac{\sin\frac{1}{2}(A+B)\sin\frac{1}{2}C - \cos\frac{1}{2}(A+B)\cos\frac{1}{2}C}{\sin\frac{1}{2}(A+B)\cos\frac{1}{2}C + \cos\frac{1}{2}(A+B)\sin\frac{1}{2}C}.$$

Substituting for $\sin\frac{1}{2}(A+B)$ and $\cos\frac{1}{2}(A+B)$ their values from (12), we have

$$\tan\tfrac{1}{2}E = \frac{\sin\frac{1}{2}C\cos\frac{1}{2}C[\cos\frac{1}{2}(a-b) - \cos\frac{1}{2}(a+b)]}{\cos\frac{1}{2}(a-b)\cos^2\frac{1}{2}C + \cos\frac{1}{2}(a+b)\sin^2\frac{1}{2}C},$$

which readily reduces to

$$\tan \tfrac{1}{2} E = \frac{\tan \frac{1}{2} a \tan \frac{1}{2} b \sin C}{1 + \tan \frac{1}{2} a \tan \frac{1}{2} b \cos C}. \tag{20}$$

35. Plane and Spherical Triangle Formulas Compared. The student will have observed that there is a striking resemblance between the formulas relating to plane triangles and certain of the formulas of the present chapter. In the table below are arranged

Plane Triangles	Spherical Triangles
I. *Law of Sines*	I. *Law of Sines*
$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$	$\dfrac{\sin a}{\sin A} = \dfrac{\sin b}{\sin B} = \dfrac{\sin c}{\sin C}$
II. *Law of Cosines*	II. *Law of Cosines*
$a^2 = b^2 + c^2 - 2bc \cos A$	$\cos a = \cos b \cos c + \sin b \sin c \cos A$
III. *Double Formulas*	III. *Delambre's Proportions*
$\sin \frac{1}{2}(A - B) \cdot \frac{1}{2} c = \frac{1}{2}(a - b) \cos \frac{1}{2} C$	$\sin \frac{1}{2}(A - B) \sin \frac{1}{2} c = \sin \frac{1}{2}(a - b) \cos \frac{1}{2} C$
$\cos \frac{1}{2}(A - B) \cdot \frac{1}{2} c = \frac{1}{2}(a + b) \sin \frac{1}{2} C$	$\cos \frac{1}{2}(A - B) \sin \frac{1}{2} c = \sin \frac{1}{2}(a + b) \sin \frac{1}{2} C$
IV. *Law of Tangents*	IV. *Law of Tangents*
$\dfrac{\frac{1}{2}(a + b)}{\frac{1}{2}(a - b)} = \dfrac{\tan \frac{1}{2}(A + B)}{\tan \frac{1}{2}(A - B)}$	$\dfrac{\tan \frac{1}{2}(a + b)}{\tan \frac{1}{2}(a - b)} = \dfrac{\tan \frac{1}{2}(A + B)}{\tan \frac{1}{2}(A - B)}$
V. *Half-angle Formulas*	V. *Half-angle Formulas*
$\sin \frac{1}{2} A = \sqrt{\dfrac{(s - b)(s - c)}{bc}}$	$\sin \frac{1}{2} A = \sqrt{\dfrac{\sin (s - b) \sin (s - c)}{\sin b \sin c}}$
$\cos \frac{1}{2} A = \sqrt{\dfrac{s(s - a)}{bc}}$	$\cos \frac{1}{2} A = \sqrt{\dfrac{\sin s \sin (s - a)}{\sin b \sin c}}$
$\tan \frac{1}{2} A = \dfrac{k}{s - a}$	$\tan \frac{1}{2} A = \dfrac{\tan k}{\sin (s - a)}$
$k = \sqrt{\dfrac{(s - a)(s - b)(s - c)}{s}}$	$\tan k = \sqrt{\dfrac{\sin (s - a) \sin (s - b) \sin (s - c)}{\sin s}}$
VI. *Area*	VI. *Area*
$\frac{1}{4} T = \sqrt{\dfrac{s}{2} \cdot \dfrac{s - a}{2} \cdot \dfrac{s - b}{2} \cdot \dfrac{s - c}{2}}$	$\tan \frac{1}{4} E = \sqrt{\tan \dfrac{s}{2} \tan \dfrac{s - a}{2} \tan \dfrac{s - b}{2} \tan \dfrac{s - c}{2}}$
	$T = r^2 E$

in parallel columns the more important formulas for plane triangles and the corresponding formulas for spherical triangles. The form of some of the formulas for plane triangles has been slightly changed in order to manifest the resemblance in the most striking manner.

36. Derivation of Formulas for Plane Triangles from Those of Spherical Triangles. We will now show that the resemblance between the two sets of formulas is not accidental but is due to a definite relation between plane and spherical triangles. If the vertices of a spherical triangle remain fixed while the radius (r) of the sphere on which the triangle is situated is indefinitely increased, the spherical triangle will approach as a limit the plane triangle having the same vertices. Consequently, for the limit $r = \infty$, the formulas for the spherical triangle must reduce to those for the plane triangle.

Let a', b', c' represent the sides of the spherical triangle expressed in radians, then $a' = a/r$, $b' = b/r$, $c' = c/r$, where a, b, c represent the actual lengths of the sides (Pl. Trig., Art. 90). Also by Pl. Trig., Art. 176, we have

$$\sin a' = \frac{a}{r} - \frac{a^3}{3!\,r^3} + \text{etc.}, \quad \cos a' = 1 - \frac{a^2}{2!\,r^2} + \frac{a^4}{4!\,r^4} - \text{etc.},$$

$$\tan a' = \frac{a}{r} + \frac{a^3}{3\,r^3} + \text{etc.}$$

and similar expressions for $\sin b'$, $\sin c'$, etc.

These expansions involve the radius of the sphere. If now we substitute these expansions in any formula relating to spherical triangles and evaluate the resulting expression for $r = \infty$, the resulting formula will express the corresponding relation between the sides and angles of the plane triangle. We will illustrate the method by some examples.

(*a*) *The Law of Sines.*

$$\frac{\sin A}{\sin B} = \frac{\sin a'}{\sin b'} = \frac{\sin a/r}{\sin b/r} = \frac{\dfrac{a}{r} - \dfrac{a^3}{3!\,r^3} + \text{etc.},}{\dfrac{b}{r} - \dfrac{b^3}{3!\,r^3} + \text{etc.},}.$$

Multiplying both numerator and denominator of the expression on the right by r, and making r infinite, we obtain

$$\frac{\sin A}{\sin B} = \frac{a}{b}, \text{ the law of sines for plane triangles.}$$

(*b*) *The Law of Cosines.*

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A$$

or

$$\cos\frac{a}{r} = \cos\frac{b}{r}\cos\frac{c}{r} + \sin\frac{b}{r}\sin\frac{c}{r}\cos A,$$

hence

$$1 - \frac{a^2}{2!\,r^2} + \frac{a^4}{4!\,r^4} + = \left(1 - \frac{b^2}{2!\,r^2} + \frac{b^4}{4!\,r^4} +\right)\left(1 - \frac{c^2}{2!\,r^2} + \frac{c^2}{4!\,r^4} +\right)$$
$$+ \left(\frac{b}{r} - \frac{b^3}{3!\,r^3} +\right)\left(\frac{c}{r} - \frac{c^3}{3!\,r^3} +\right)\cos A.$$

If we multiply both sides of the equation by $-2r^2$, drop the terms which are common to both sides of the equation, and then make r infinite, we have

$a^2 = b^2 + c^2 - 2\,bc \cos A$, the law of cosines for plane triangles.

(*c*) *The Law of Tangents.*

$$\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)} = \frac{\tan\frac{1}{2}(a'+b')}{\tan\frac{1}{2}(a'-b')} = \frac{\tan\dfrac{a+b}{2r}}{\tan\dfrac{a-b}{2r}} = \frac{\dfrac{a+b}{2r} + \dfrac{(a+b)^3}{3(2r)^3} +}{\dfrac{a-b}{2r} + \dfrac{(a-b)^3}{3(2r)^3} +}.$$

Multiplying both numerator and denominator on the right by $2r$ and making r infinite, we have

$$\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)} = \frac{a+b}{a-b},$$ the law of tangents for plane triangles.

(*d*) *Area of a Triangle.* As a final example we will deduce *Hero's* formula for the area of a plane triangle from *Lhuillier's* formula for the spherical excess.

Denote $a' + b' + c'$ by $2s'$, then $s' = s/r$, $s' - a' = (s-a)/r$, $s' - b' = (s-b)/r$, etc., and we have from Lhuillier's formula

$$\frac{E}{4} + \frac{E^3}{3\cdot 4^3} + =$$
$$\sqrt{\left(\frac{s}{2r} + \frac{s^3}{3(2r)^3} +\right)\left(\frac{s-a}{2r} + \frac{(s-a)^3}{3(2r)^3} +\right)\left(\frac{s-b}{2r} + \frac{(s-b)^3}{3(2r)^3} +\right)\left(\frac{s-c}{2r} + \frac{(s-c)^3}{3(2r)^3} +\right)}.$$

Multiplying through by $4r^2$ gives

$$r^2E + \frac{r^2E\cdot E^2}{3\cdot 4^2} + =$$
$$\sqrt{\left(s + \frac{s^3}{12\,r^2} +\right)\left(s - a + \frac{(s-a)^3}{12\,r^2} +\right)\left(s - b + \frac{(s-b)^3}{12\,r^2} +\right)\left(s - c + \frac{(s-c)^3}{12\,r^2} +\right)}.$$

Now as r approaches infinity, E approaches 0, r^2E remains equal to the area of the triangle, hence in the limit

$$r^2E = T = \sqrt{s(s-a)(s-b)(s-c)}$$

which is Hero's formula.

EXERCISE 9

1. Derive the second of the formulas (12).

2. Derive the third and fourth of the formulas (12) by applying the first and second to the co-lunar triangle $AB'C$.

3. Derive the fourth of the formulas (13) by applying the third to the co-lunar triangle $AB'C$.

4. Derive the fourth of the formulas (13) by applying the second to the polar triangle.

5. Show that the area of the co-lunar triangle $A'BC, AB'C, ABC'$ is $r^2(2A - E)$, $r^2(2B - E)$, $r^2(2C - E)$, respectively, where E is the spherical excess of the triangle ABC.

6. Prove that

$$\sin(s-a) + \sin(s-b) + \sin(s-c) - \sin s = 4 \sin \tfrac{1}{2}a \sin \tfrac{1}{2}b \sin \tfrac{1}{2}c.$$

7. If S, S_A, S_B, S_C denote half the sums of the angles of a triangle and its three co-lunars respectively, prove that

$$S + S_A + S_B + S_C = 3\pi.$$

8. If E, E_A, E_B, E_C denote the spherical excesses of a triangle and its three co-lunars respectively, show that $E + E_A + E_B + E_C = 2\pi$, and hence that the sum of the area of these triangles is equal to half the area of the sphere.

9. Deduce the double formula for plane triangles from Delambre's formulas for spherical triangles.

10. Deduce the half-angle formulas for plane triangles from the corresponding formulas for spherical triangles.

11. From the formula $\cos c = \cos a \cos b$ for right spherical triangles deduce the formula $c^2 = a^2 + b^2$ for plane right triangles.

12. If K, K_A, K_B, K_C denote the arcual radii of a triangle and its three co-lunars, show that $\tan K \cot K_A \cot K_B \cot K_C = \cos^2 S$.

CHAPTER IV

SOLUTION OF OBLIQUE SPHERICAL TRIANGLES

37. Preliminary Observations. In Art. 23 it was shown that every spherical triangle may be solved by the method of right triangles. Again every spherical triangle may be solved by means of the fundamental relations of Art. 27, as was shown in Art. 29. The purpose of the present chapter is to present the most approved methods, which, though based on apparently more complicated formulas, require, as a rule, the least possible amount of computation, and are, therefore, commonly employed by computers.

The computer will do well to observe the following points:

(*a*) The arrangement of the work should be orderly and methodical. A complete schedule for the tabular work should be made out before the tables are used (Pl. Trig., Art. 70).

(*b*) It will be well to letter the given parts as in the illustrations which follow. Thus if the given parts are two sides and the included angle, call the larger of the two sides a, the other b, and the angle C. This is easier than to rewrite the formulas so as to involve other letters.

(*c*) Remember that a small angle cannot be accurately found from its cosine, nor an angle near 90° from its sine. (Pl. Trig., Art. 21.) Usually there is a choice of formulas which will enable us to avoid any inaccuracies arising from this source.

(*d*) Remember also that the answer cannot be more accurate than the least accurate of the given parts. It is a false show of accuracy to compute the answer to the nearest second when one or more of the given parts have a lesser accuracy. (Pl. Trig., Art. 44, 19.)

(*e*) No result can be relied upon unless it has been checked. When the answer is given, that may be looked upon as a check, in all other cases the computer must provide a check of his own.

38. Case I. Given the Three Sides, *a, b, c*.

Solution.

1. To find A, B, C. Use the half-angle formulas (8).
2. Check. Use the law of sines.

Note. If one angle only is required it is better to use (6) or (7).

EXAMPLE.

Given

$a = 123° 34' 45''$,
$b = 75° 56' 33''$,
$c = 105° 00' 18''$.

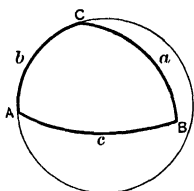



Fig. 39.

To find

$A = 121° 32' 41''$,
$B = 82° 52' 53''$,
$C = 98° 51' 55''$.

Solution.

1. To find A, B, and C.

$$\tan \tfrac{1}{2} A = \frac{\tan k}{\sin (s-a)}, \quad \tan \tfrac{1}{2} B = \frac{\tan k}{\sin (s-b)}, \quad \tan \tfrac{1}{2} C = \frac{\tan k}{\sin (s-c)},$$

$$\tan k = \sqrt{\frac{\sin (s-a) \sin (s-b) \sin (s-c)}{\sin s}}, \quad s = \frac{a+b+c}{2}.$$

$a = 123° 34' 45''$ | $\log \sin (s-a) = 9.68122$
$b = 75° 56' 33''$ | $\log \sin (s-b) = 9.98751$
$c = 105° 00' 18''$ | $\log \sin (s-c) = 9.86594$
$2s = 304° 31' 36''$ | $\text{colog} \sin s = 0.33215$
| $\log \tan^2 k = 9.86682$
$s = 152° 15' 48''$ | $\log \tan k = 9.93341$
$s-a = 28° 41' 03''$
$s-b = 76° 19' 15''$ | $\log \tan \frac{1}{2} A = 0.25219$
$s-c = 47° 15' 30''$ | $\log \tan \frac{1}{2} B = 9.94590$
$2s = 152° 15' 48''$ (check) | $\log \tan \frac{1}{2} C = 0.06747$

$\frac{1}{2} A = 60° 46' 20.7''$ | $A = 121° 32' 41''$,
$\frac{1}{2} B = 41° 26' 26.4''$ | $B = 82° 52' 53''$,
$\frac{1}{2} C = 49° 25' 57.7''$ | $C = 98° 51' 55''$.

2. Check.

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

$\log \sin a = 9.92071$ | $\log \sin b = 9.98680$ | $\log \sin c = 9.98493$
$\log \sin A = 9.93056$ | $\log \sin B = 9.99664$ | $\log \sin C = 9.99478$
9.99015 | 9.99016 | 9.99015

EXERCISE 10

Solve the following oblique triangles:

1. Given $a = 72° 16'$, $b = 80° 44'$, $c = 41° 18'$.
Ans. $A = 73° 38'$, $B = 96° 12'$, $C = 41° 40'$.

2. Given $a = 109° 45'$, $b = 73° 56'$, $c = 54° 32'$.

3. Given $a = 105° 06.8'$, $b = 93° 39.9'$, $c = 50° 20.3'$.
Ans. $A = 106° 38.0'$, $B = 82° 04.4'$, $C = 49° 49.2'$.

4. Given $a = 27° 43.8'$, $b = 49° 36.8'$, $c = 55° 19.7'$.

5. Given $a = 120° 22' 40''$ $b = 111° 34' 27''$, $c = 96° 28' 35''$.
Ans. $A = 126° 18' 42''$, $B = 119° 42' 08''$, $C = 111° 51' 42''$.

6. Given $a = 20° 45' 23''$, $b = 55° 56' 56''$, $c = 67° 25' 54''$.

7. Given $a = 131° 35' 04''$, $b = 108° 30' 14''$, $c = 84° 46' 34''$, $A = 132° 14' 21''$. Find B and C.

8. Given $a = 35° 30' 24''$, $b = 38° 57' 12''$, $c = 56° 15' 43''$. Find $B = 47° 37' 21''$.

39. Case II. Given the Three Angles, *A*, *B*, *C*.

Solution.

1. To find a, b, c. Use the half-angle formula (11).

2. Check. Use the law of sines.

Note. If one side only is required it is better to use (9) or (10).

EXAMPLE.

Given	To find
$A = 121° 32' 41''$,	$a = 123° 34' 46''$,
$B = 82° 52' 53''$,	$b = 75° 56' 32''$,
$C = 98° 51' 55''$.	$c = 105° 00' 18''$.

Solution.

1. To find a, b, c.

$$\tan \tfrac{1}{2} a = \tan K \cos (S - A), \quad \tan \tfrac{1}{2} b = \tan K \cos (S - B), \quad \tan \tfrac{1}{2} c = \tan K \cos (S - C),$$

$$\tan K = \sqrt{\frac{-\cos S}{\cos (S - A) \cos (S - B) \cos (S - C)}}, \quad S = \frac{A + B + C}{2}.$$

$$\begin{array}{rlrl}
A = & 121° 32' 41'' & & \\
B = & 82° 52' 53'' & S - A = & 30° 06' 03.5'' \\
C = & 98° 51' 55'' & S - B = & 68° 45' 51.5'' \\
2S = & 303° 17' 29'' & S - C = & 52° 46' 49.5'' \\
S = & 151° 38' 44.5'' & S = & 151° 38' 44.5'' \text{ (check)}
\end{array}$$

		$\log(-\cos S)$	$= 9.94450$
$\log\cos(S-A)$	$= 9.93709$	$\text{colog}\cos(S-A)$	$= 0.06291$
$\log\cos(S-B)$	$= 9.55896$	$\text{colog}\cos(S-B)$	$= 0.44104$
$\log\cos(S-C)$	$= 9.78166$	$\text{colog}\cos(S-C)$	$= 0.21834$
		$\log\tan^2 K$	$= 0.66679$
$\log\tan K$	$= 0.33340$	$\log\tan K$	$= 0.33340$
$\log\tan\frac{1}{2}a$	$= 0.27049$	$\frac{1}{2}a$	$= 61^\circ\,47'\,23''$
$\log\tan\frac{1}{2}b$	$= 9.89236$	$\frac{1}{2}b$	$= 37^\circ\,58'\,16''$
$\log\tan\frac{1}{2}c$	$= 0.11506$	$\frac{1}{2}c$	$= 52^\circ\,30'\,09''$

$$a = 123^\circ\,34'\,46'',\quad b = 75^\circ\,56'\,32'',\quad c = 105^\circ\,00'\,18''.$$

2. Check.

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

$\log\sin A$	$= 9.93056$	$\log\sin B$	$= 9.99664$	$\log\sin C$	$= 9.99478$
$\log\sin a$	$= 9.92071$	$\log\sin b$	$= 9.98680$	$\log\sin c$	$= 9.98493$
	0.00985		0.00984		0.00985

Note. Since the sum of the angles of a spherical triangle is always between 180° and 540°, S is necessarily between 90° and 270°, hence, $\cos S$ is always negative and $-\cos S$ positive.

Exercise 11

Solve the following triangles:

1. Given $A = 74^\circ\,40'$, $B = 67^\circ\,30'$, $C = 49^\circ\,50'$.
   Ans. $a = 43^\circ\,36'$, $b = 41^\circ\,21'$, $c = 33^\circ\,07'$.
2. Given $A = 125^\circ\,54'$, $B = 55^\circ\,35'$, $C = 45^\circ\,05'$.
3. Given $A = 46^\circ\,59.3'$, $B = 122^\circ\,32.6'$, $C = 139^\circ\,00.3'$.
   Ans. $a = 59^\circ\,27.4'$, $b = 117^\circ\,06.2'$, $c = 123^\circ\,20.0'$.
4. Given $A = 47^\circ\,34.6'$, $B = 74^\circ\,54.7'$, $C = 77^\circ\,24.5'$.
5. Given $A = 59^\circ\,55'\,10''$, $B = 85^\circ\,36'\,50''$, $C = 59^\circ\,55'\,10''$.
   Ans. $a = 51^\circ\,17'\,31''$, $b = 64^\circ\,02'\,47''$, $c = 51^\circ\,17'\,31''$.
6. Given $A = 109^\circ\,35'\,56''$, $B = 111^\circ\,23'\,06''$, $C = 86^\circ\,49'\,19''$.
7. Given $A = 15^\circ\,38'\,06''$, $B = 16^\circ\,06'\,22''$, $C = 159^\circ\,44'\,26''$.
   Find b. *Ans.* $b = 52^\circ\,05'\,54''$.
8. Given $A = 50^\circ\,45'\,23''$, $B = 58^\circ\,01'\,10''$, $C = 87^\circ\,17'\,00''$.
   Find C.

40. Case III. Given Two Sides and the Included Angle, *a, b, C*.

Solution.

1. To find A and B. First find $\frac{1}{2}(A+B)$ and $\frac{1}{2}(A-B)$ by the first two of Napier's proportions (Art. 33), then

$$A = \tfrac{1}{2}(A+B) + \tfrac{1}{2}(A-B), \quad B = \tfrac{1}{2}(A+B) - \tfrac{1}{2}(A-B).$$

2. To find c. Use either one of Delambre's proportions (Art. 32).
3. Check. Use the law of sines (Art. 24).

EXAMPLE.

Given

$a = 110° 30' 24''$,
$b = 36° 47' 36''$,
$C = 135° 12' 12''$.

C
a
b
B
c
A

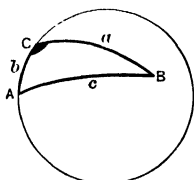



Fig. 40.

To find

$A = 63° 57' 39''$,
$B = 35° 04' 03''$,
$c = 132° 44' 08''$.

Solution.

1. To find A and B.

$$\tan\tfrac{1}{2}(A+B) = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C.$$

$$\tan\tfrac{1}{2}(A-B) = \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C.$$

$\frac{1}{2}(a-b) = 36° 51' 24''$, $\frac{1}{2}(a+b) = 73° 39' 00''$, $\frac{1}{2}C = 67° 36' 06''$.

$\log\cos\frac{1}{2}(a-b)$	$= 9.90316$	$\log\sin\frac{1}{2}(a-b)$	$= 9.77802$
$\operatorname{colog}\cos\frac{1}{2}(a+b)$	$= 0.55052$	$\operatorname{colog}\sin\frac{1}{2}(a+b)$	$= 0.01793$
$\log\cot\frac{1}{2}C$	$= 9.61504$	$\log\cot\frac{1}{2}C$	$= 9.61504$
$\log\tan\frac{1}{2}(A+B)$	$= 0.06872$	$\log\tan\frac{1}{2}(A-B)$	$= 9.41099$
$\frac{1}{2}(A+B)$	$= 49° 30' 51''$	$\frac{1}{2}(A-B)$	$= 14° 26' 48''$
A	$= 63° 57' 39''$.	B	$= 35° 04' 03''$.

2. To find c.

$$\cos\tfrac{1}{2}c = \frac{\cos\frac{1}{2}(a-b)}{\sin\frac{1}{2}(A+B)}\cos\tfrac{1}{2}C.$$

$\log\cos\frac{1}{2}(a-b)$	$= 9.90316$
$\operatorname{colog}\sin\frac{1}{2}(A+B)$	$= 0.11886$
$\log\cos\frac{1}{2}C$	$= 9.58098$
$\log\cos\frac{1}{2}c$	$= 9.60300$

3. Check.

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

$\log\sin a$	$= 9.97157$
$\log\sin A$	$= 9.95352$
	0.01805

$$\tfrac{1}{2} c = 66^\circ\ 22'\ 04''$$
$$c = 132^\circ\ 44'\ 08''.$$

$$\begin{aligned} \log \sin b &= 9.77738 \\ \log \sin B &= \underline{9.75932} \\ &\quad 0.01806 \\ \log \sin c &= 9.86598 \\ \log \sin C &= \underline{9.84794} \\ &\quad 0.01804 \end{aligned}$$

We might have found c from the third or fourth of Napier's proportions but this would have required us to look up one more logarithm.

Exercise 12

Solve the following oblique triangles:

1. Given $a = 140^\circ\ 38'$, $b = 130^\circ\ 28'$, $C = 150^\circ\ 34'$.
   Ans. $A = 161^\circ\ 47'$, $B = 157^\circ\ 58'$, $c = 85^\circ\ 20'$.
2. Given $a = 103^\circ\ 44'$, $b = 64^\circ\ 12'$, $C = 98^\circ\ 33'$.
3. Given $a = 156^\circ\ 12.2'$, $b = 112^\circ\ 48.6'$, $C = 76^\circ\ 32.4'$.
   Ans. $A = 154^\circ\ 04.1'$, $B = 87^\circ\ 27.1'$, $c = 63^\circ\ 48.8'$.
4. Given $a = 27^\circ\ 45.5'$, $b = 22^\circ\ 56.7'$, $C = 156^\circ\ 15.9'$.
5. Given $a = 88^\circ\ 12'\ 20''$, $b = 124^\circ\ 07'\ 17''$, $C = 50^\circ\ 02'\ 02''$.
   Ans. $A = 63^\circ\ 15'\ 10''$, $B = 132^\circ\ 17'\ 59''$, $c = 59^\circ\ 04'\ 25''$.
6. Given $a = 111^\circ\ 11'\ 12''$, $b = 137^\circ\ 56'\ 56''$, $C = 23^\circ\ 15'\ 48''$.
7. Given $b = 68^\circ\ 12'\ 58''$, $c = 80^\circ\ 14'\ 41''$, $A = 17^\circ\ 20'\ 54''$.
   Ans. $B = 52^\circ\ 05'\ 54''$, $C = 123^\circ\ 07'\ 37''$, $a = 20^\circ\ 32'\ 33''$.
8. Given $a = 56^\circ\ 56'\ 56''$, $c = 156^\circ\ 56'\ 56''$, $B = 94^\circ\ 45'\ 45''$.

41. Case IV. Given Two Angles and the Included Side, A, B, c.

Solution.

1. To find a and b. First find $\frac{1}{2}(a + b)$ and $\frac{1}{2}(a - b)$ by the last two of Napier's proportions (Art. 33), then

$$a = \tfrac{1}{2}(a + b) + \tfrac{1}{2}(a - b), \quad b = \tfrac{1}{2}(a + b) - \tfrac{1}{2}(a - b).$$

2. To find c. Use either one of Delambre's proportions (Art. 32).
3. Check. Use the law of sines.

Example.

Given	To find
$A = 63^\circ\ 57'\ 39''$,	$a = 110^\circ\ 30'\ 23''$,
$B = 35^\circ\ 04'\ 03''$,	$b = 36^\circ\ 47'\ 37''$,
$c = 132^\circ\ 44'\ 08''$.	$C = 135^\circ\ 12'\ 15''$.

Solution.

1. To find a and b.

$$\tan \tfrac{1}{2}(a+b) = \frac{\cos \tfrac{1}{2}(A-B)}{\cos \tfrac{1}{2}(A+B)} \tan \tfrac{1}{2} c.$$

$$\tan \tfrac{1}{2}(a-b) = \frac{\sin \tfrac{1}{2}(A-B)}{\sin \tfrac{1}{2}(A+B)} \tan \tfrac{1}{2} c.$$

$\tfrac{1}{2}(A-B) = 14^\circ\, 26'\, 48''$, $\tfrac{1}{2}(A+B) = 49^\circ\, 30'\, 51''$, $\tfrac{1}{2} c = 66^\circ\, 22'\, 04''$.

$\log \cos \tfrac{1}{2}(A-B) = 9.98605$
$\text{colog} \cos \tfrac{1}{2}(A+B) = 0.18758$
$\log \tan \tfrac{1}{2} c = 0.35896$
$\log \tan \tfrac{1}{2}(a+b) = 0.53259$
$\tfrac{1}{2}(a+b) = 73^\circ\, 39'\, 00''$
$a = 110^\circ\, 30'\, 23''.$

$\log \sin \tfrac{1}{2}(A-B) = 9.39702$
$\text{colog} \sin \tfrac{1}{2}(A+B) = 0.11887$
$\log \tan \tfrac{1}{2} c = 0.35896$
$\log \tan \tfrac{1}{2}(a-b) = 9.87485$
$\tfrac{1}{2}(a-b) = 36^\circ\, 51'\, 23''$
$b = 36^\circ\, 47'\, 37''.$

2. To find C.

$$\cos \tfrac{1}{2} C = \frac{\sin \tfrac{1}{2}(A-B)}{\sin \tfrac{1}{2}(a-b)} \sin \tfrac{1}{2} c.$$

$\log \sin \tfrac{1}{2}(A-B) = 9.39702$
$\text{colog} \sin \tfrac{1}{2}(a-b) = 0.22199$
$\log \sin \tfrac{1}{2} c = 9.96196$
$\log \cos \tfrac{1}{2} C = 9.58097$
$\tfrac{1}{2} C = 67^\circ\, 36'\, 07.7''$
$C = 135^\circ\, 12'\, 15''$

3. Check.

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

$\log \sin A = 9.95352$
$\log \sin a = 9.97157$
9.98195
$\log \sin B = 9.75932$
$\log \sin b = 9.77738$
9.98194
$\log \sin C = 9.84793$
$\log \sin c = 9.86598$
9.98195

Exercise 13

Solve the following triangles:

1. Given $A = 67^\circ\, 30'$, $B = 45^\circ\, 50'$, $c = 74^\circ\, 20'$.
   Ans. $a = 63^\circ\, 15'$, $b = 53^\circ\, 46'$, $C = 52^\circ\, 27'$.
2. Given $A = 126^\circ\, 45'$, $B = 49^\circ\, 52'$, $c = 80^\circ\, 01'$.
3. Given $B = 140^\circ\, 43.2'$, $C = 100^\circ\, 04.6'$, $a = 60^\circ\, 43.6'$.
   Ans. $b = 145^\circ\, 55.2'$, $c = 119^\circ\, 22.6'$, $A = 80^\circ\, 14.8'$.
4. Given $C = 139^\circ\, 25.8'$, $A = 13^\circ\, 56.9'$, $b = 29^\circ\, 00.8'$.
5. Given $A = 153^\circ\, 17'\, 06''$, $B = 78^\circ\, 43'\, 32''$, $c = 86^\circ\, 15'\, 15''$.
   Ans. $a = 88^\circ\, 12'\, 19''$, $b = 78^\circ\, 15'\, 41''$, $C = 152^\circ\, 43'\, 52''$.
6. Given $a = 50^\circ\, 34'\, 56''$, $B = 124^\circ\, 10'\, 10''$, $C = 83^\circ\, 25'\, 25''$.

42. Case V. Given Two Sides and the Angle Opposite One of Them, *a, b, A.*

In this case there may be two solutions (see Art. 11).

1. To find B. Use the law of sines, $\sin B = \dfrac{\sin b \sin A}{\sin a}$.

Since B is found from its sine it will in general have two values whose sum is 180°. $\sin B = \dfrac{\sin b \sin A}{\sin a} \lesseqqgtr 1$, according as $\sin b \sin A \lesseqqgtr \sin a$, hence B has two values, one value (90°), or no real value, according as $\sin b \sin A \lesseqqgtr \sin a$.

2. To find C. From the second of Napier's proportions

$$\tan \tfrac{1}{2} C = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \cot \tfrac{1}{2}(A - B).$$

Since C is less than 180°, $\tan \frac{1}{2} C$ must be positive. Now $a + b$ is always less than 360°, therefore $\sin \frac{1}{2}(a + b)$ is always positive, hence in order that $\tan \frac{1}{2} C$ may be positive $\sin \frac{1}{2}(a - b)$ and $\cot \frac{1}{2}(A - B)$ must have like signs. Now $\frac{1}{2}(a - b)$ and $\frac{1}{2}(A - B)$ are each numerically less than 90°, hence in order that $\sin \frac{1}{2}(a - b)$ and $\cot \frac{1}{2}(A - B)$ may have like signs, $\frac{1}{2}(a - b)$ and $\frac{1}{2}(A - B)$ and consequently $a - b$ and $A - B$ must have like signs. If both values of B satisfy this condition there are two solutions, if only one value of B satisfies this condition there is only one solution, if neither value of B satisfies this condition there is no solution.

3. To find c. From the fourth of Napier's proportions

$$\tan \tfrac{1}{2} c = \frac{\sin \frac{1}{2}(A + B)}{\sin \frac{1}{2}(A - B)} \tan \tfrac{1}{2}(a - b).$$

4. Check. Use the law of sines, or any other formula involving B, C, and c, which has not been previously used.

The foregoing considerations regarding the number of admissible solutions may be summed up into the following:

Rule.

a. If $\sin a < \sin b \sin A$, *there is no solution.*
b. If $\sin a = \sin b \sin A$, *there is one solution,* $B = 90°$.
c. If $\sin a > \sin b \sin A$, *each of the two values of* B *which gives* $A - B$ *and* $a - b$ *like signs yields a solution.*

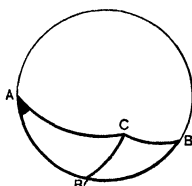
EXAMPLE.

Given.	To find.
$a = 62° 15'$,	$B = 62° 25'$,
$b = 103° 19'$,	$C = 155° 43'$,
$A = 53° 43'$.	$c = 153° 10'$,
	$B' = 117° 35'$,
	$C' = 59° 08'$,
	$C' = 70° 27'$.

Fig. 41.

Solution.

1. To find B.

$$\sin B = \frac{\sin b \sin A}{\sin a}.$$

$$\begin{aligned}
\log \sin b &= 9.98816\\
\log \sin A &= 9.90639\\
\operatorname{colog} \sin a &= 0.05306\\
\log \sin B &= 9.94761
\end{aligned}$$

$$B = 62° 25' \quad \text{or} \quad B' = 117° 35'.$$

2. To find C.

$$\tan \tfrac{1}{2} C = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \cot \tfrac{1}{2}(A-B).$$

3. To find c.

$$\tan \tfrac{1}{2} c = \frac{\sin \frac{1}{2}(A+B)}{\sin \frac{1}{2}(A-B)} \tan \tfrac{1}{2}(a-b).$$

$\frac{1}{2}(a+b) = 82° 47'$, $\quad \frac{1}{2}(A+B) = 58° 04'$, $\quad \frac{1}{2}(A+B') = 85° 39'$,
$\frac{1}{2}(a-b) = -20° 32'$, $\quad \frac{1}{2}(A-B) = -4° 21'$, $\quad \frac{1}{2}(A-B') = -31° 56'$.

Since the signs of $a - b$ and $A - B$ are alike for both values of B there are two solutions.

$$\begin{aligned}
\log \sin \tfrac{1}{2}(a-b) &= 9.54500n\\
\operatorname{colog} \sin \tfrac{1}{2}(a+b) &= 0.00345\\
\log \cot \tfrac{1}{2}(A-B) &= 1.11880n\\
\log \cot \tfrac{1}{2}(A-B') &= 0.20534n\\
\log \tan \tfrac{1}{2} C &= 0.66725\\
\log \tan \tfrac{1}{2} C' &= 9.75379
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2} C &= 77° 51.5'.\\
\tfrac{1}{2} C' &= 29° 33.9'.\\
C &= 155° 43.0'.\\
C' &= 59° 07.8'.
\end{aligned}$$

$$\begin{aligned}
\log \sin \tfrac{1}{2}(A+B) &= 9.92874\\
\operatorname{colog} \sin \tfrac{1}{2}(A-B) &= 1.12005n\\
\log \tan \tfrac{1}{2}(a-b) &= 9.57351n\\
\log \sin \tfrac{1}{2}(A+B') &= 9.99875\\
\operatorname{colog} \sin \tfrac{1}{2}(A-B') &= 0.27660n\\
\log \tan \tfrac{1}{2} c &= 0.62230\\
\log \tan \tfrac{1}{2} c' &= 9.84886
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2} c &= 76° 34.8'.\\
\tfrac{1}{2} c' &= 35° 13.5'.\\
c &= 153° 09.6'.\\
c' &= 70° 27.0'.
\end{aligned}$$

4. Check.

$$\frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} = \frac{\sin c'}{\sin C'}.$$

$\log \sin b = 9.98816$	$\log \sin c = 9.65466$	$\log \sin c' = 9.97421$
$\log \sin B = 9.94761$	$\log \sin C = 9.61411$	$\log \sin C' = 9.93366$
0.04055	0.04055	0.04055

EXERCISE 14

Solve the following triangles:

1. $a = 56° 40'$, $b = 30° 50'$, $A = 103° 40'$.
Ans. $B = 36° 36'$, $C = 52° 00'$, $c = 42° 39'$.

2. $b = 44° 45'$, $c = 49° 35'$, $B = 58° 56'$. (Two solutions.)

3. $a = 148° 34.4'$, $b = 142° 11.6'$, $A = 153° 17.6'$.
Ans. $B = 31° 53.7'$, $C = 6° 17.6'$, $c = 7° 18.3'$;
$B' = 148° 06.3'$, $C' = 130° 21.4'$, $c' = 62° 08.8'$.

4. $a = 41° 25.8'$, $b = 19° 57.9'$, $A = 62° 09.5'$. (One solution.)

5. $a = 67° 12' 20''$, $b = 48° 45' 40''$, $B = 42° 20' 30''$.
Ans. $A = 55° 39' 57''$, $C = 116° 34' 18''$, $c = 93° 08' 10''$;
$A' = 124° 20' 03''$, $C' = 24° 32' 15''$, $c' = 27° 37' 20''$.

6. $a = 38° 10' 10''$, $b = 24° 56' 45''$, $B = 65° 25' 00''$. (No solution.)

43. Case VI. Given Two Angles and the Side Opposite One of Them, A, B, a.

As in Case V so here there may be two solutions. (See Art. 11.)

1. To find b. Use the law of sines,

$$\sin b = \frac{\sin B \sin a}{\sin A}.$$

2. To find c. From the fourth of Napier's proportions,

$$\cot \tfrac{1}{2} c = \frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} \cot \tfrac{1}{2}(a - b).$$

3. To find C. From the second of Napier's proportions,

$$\cot \tfrac{1}{2} C = \frac{\sin \frac{1}{2}(a + b)}{\sin \frac{1}{2}(a - b)} \tan \tfrac{1}{2}(A - B).$$

4. Check. Use the law of sines, or any other formula involving b, c, and C, which has not been previously used.

To determine the number of solutions we have the following rule which is based upon a process of reasoning exactly analogous to that employed in establishing the corresponding rule in Case V.

Rule.

a. If $\sin A < \sin B \sin a$, there is no solution.

b. If $\sin A = \sin B \sin a$, there is one solution, $b = 90°$.

c. If $\sin A > \sin B \sin a$, each of the two values of b, which gives to $a - b$ and $A - B$ like signs, yields a solution.

EXAMPLE.

Given	To find
$A = 45^\circ\, 30'$,	$b = 33^\circ\, 38'$,
$B = 37^\circ\, 22'$,	$c = 59^\circ\, 15'$,
$a = 40^\circ\, 36'$.	$C = 109^\circ\, 37'$.

Solution.

1. To find b.

$$\begin{aligned} \log\sin B &= 9.78312 \\ \log\sin a &= 9.81343 \\ \text{colog}\sin A &= 0.14676 \\ \log\sin b &= 9.74331 \\ b &= 33^\circ\, 37.5' \\ b' &= 146^\circ\, 22.5' \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}(A+B) &= 41^\circ\, 26' \\ \tfrac{1}{2}(A-B) &= 4^\circ\, 04' \\ \tfrac{1}{2}(a+b) &= 37^\circ\, 06.8' \\ \tfrac{1}{2}(a-b) &= 3^\circ\, 29.2' \\ \tfrac{1}{2}(a+b') &= 93^\circ\, 29.2' \\ \tfrac{1}{2}(a-b') &= -52^\circ\, 48.2' \end{aligned}$$

$A - B$ and $a - b'$ have unlike signs, hence b' does not yield a solution.

2. To find c.

$$\begin{aligned} \log\sin\tfrac{1}{2}(A-B) &= 8.85075 \\ \text{colog}\sin\tfrac{1}{2}(A+B) &= 0.17931 \\ \log\cot\tfrac{1}{2}(a-b) &= 1.21507 \\ \log\cot\tfrac{1}{2}c &= 0.24513 \\ \tfrac{1}{2}c &= 29^\circ\, 37.6' \\ c &= 59^\circ\, 15.2' \end{aligned}$$

3. To find C.

$$\begin{aligned} \log\sin\tfrac{1}{2}(a+b) &= 9.78060 \\ \text{colog}\sin\tfrac{1}{2}(a-b) &= 1.21588 \\ \log\tan\tfrac{1}{2}(A-B) &= 8.85185 \\ \log\cot\tfrac{1}{2}C &= 9.84833 \\ \tfrac{1}{2}C &= 54^\circ\, 48.45 \\ C &= 109^\circ\, 36.9' \end{aligned}$$

4. Check.

$$\frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

$$\begin{aligned} \log\sin B &= 9.78312 \\ \log\sin b &= 9.74331 \\ & \overline{0.03381} \end{aligned} \qquad \begin{aligned} \log\sin C &= 9.97403 \\ \log\sin c &= 9.93421 \\ & \overline{0.03382} \end{aligned}$$

EXERCISE 15

Solve the following triangles:

1. $A = 36^\circ\, 20'$, $B = 46^\circ\, 30'$, $a = 42^\circ\, 12'$.

   Ans. $b = 55^\circ\, 19'$, $c = 81^\circ\, 19'$, $C = 119^\circ\, 19'$;
   $b' = 124^\circ\, 51'$, $c' = 162^\circ\, 38'$, $C' = 164^\circ\, 44'$.

2. $A = 60^\circ\, 32'$, $B = 25^\circ\, 56'$, $a = 35^\circ\, 18'$. (One solution.)

3. $A = 73^\circ\, 11.3'$, $B = 61^\circ\, 18.2'$, $a = 46^\circ\, 45.5'$.

   Ans. $b = 41^\circ\, 52.6'$, $c = 41^\circ\, 35.1'$, $C = 60^\circ\, 42.8'$.

4. $A = 103^\circ\, 56.9'$, $B = 79^\circ\, 35.8'$, $a = 127^\circ\, 45.0'$. (Two solutions.)

5. $B = 123° 40' 20''$, $C = 159° 43' 22''$, $c = 159° 50' 05''$.
Ans. $b = 55° 52' 30''$, $a = 137° 21' 19''$, $A = 137° 04' 26''$;
$b' = 124° 07' 30''$, $a' = 65° 39' 44''$, $A' = 113° 39' 16''$.

6. $A = 70° 45' 10''$, $B = 119° 56' 56''$, $b = 79° 45' 02''$. (No solution.)

44. To Find the Area of a Spherical Triangle.

EXAMPLE.

Given $a = 124° 12' 31''$, $b = 54° 18' 16''$, $c = 97° 12' 25''$. Find the spherical excess, and hence the area of the triangle, the radius of the sphere being 3959 miles.

Solution. By Art. 34 we have

$$\tan \tfrac{1}{4} E = \sqrt{\tan \tfrac{1}{2} s \tan \tfrac{1}{2}(s-a) \tan \tfrac{1}{2}(s-b) \tan \tfrac{1}{2}(s-c)}, \quad T = \frac{\pi R^2 E°}{180°}.$$

$$\begin{array}{rl}
\tfrac{1}{2} a = & 62° 06' 15.5'' \\
\tfrac{1}{2} b = & 27° 09' 08'' \\
\tfrac{1}{2} c = & \underline{48° 36' 12.5''} \\
s = & 137° 51' 36'' \\
\tfrac{1}{2} s = & 68° 55' 48'' \\
\tfrac{1}{2}(s-a) = & 6° 49' 32.5'' \\
\tfrac{1}{2}(s-b) = & 41° 46' 40'' \\
\tfrac{1}{2}(s-c) = & \underline{20° 19' 35.5''} \\
\text{(check)} & 68° 55' 48''
\end{array}
\qquad
\begin{array}{rl}
\log \tan \tfrac{1}{2} s = & 0.41426 \\
\log \tan \tfrac{1}{2}(s-a) = & 9.07809 \\
\log \tan \tfrac{1}{2}(s-b) = & 9.95105 \\
\log \tan \tfrac{1}{2}(s-c) = & \underline{9.56871} \\
\log \tan^2 \tfrac{1}{4} E = & 9.01211 \\
\log \tan \tfrac{1}{4} E = & 9.50605 \\
\tfrac{1}{4} E° = & 17° 46' 45'' \\
E° = & 71° 07' 00''
\end{array}$$

$$\begin{array}{rl}
\log R = & 3.59759 \\
\log R^2 = & 7.19518 \\
\log \pi = & 0.49715 \\
\log E° = & 1.85197 \\
\text{colog } 180° = & \underline{7.74473} \\
\log T = & 7.28903 \\
T = & 19455 \times 10^3 \text{ square miles.}
\end{array}$$

45. Applications to Geometry.

EXERCISE 16

Right Spherical Triangles

1. The hypotenuse of an isosceles right spherical triangle is 60°. Find the length of the equal sides. *Ans.* 45°.

2. Find the relations between each two of the three distinct parts of an isosceles right spherical triangle. *Ans.* $\cos c = \cos^2 a = \cot^2 A$.

3. Show that no isosceles right spherical triangle can have its hypotenuse greater than 90° nor its acute angle less than 45°.

4. Find the altitude and angle of an equilateral spherical triangle whose side is 60°. *Ans.* Altitude = 54° 44′, Angle = 70° 32′.

5. If a is the side, A the angle, and p the altitude of an equilateral spherical triangle, show that $\sin \frac{1}{2} a \sin \frac{1}{2} A = \frac{1}{2}$, $\cos p = \dfrac{\cos a}{\cos \frac{1}{2} a}$.

6. The side of a spherical square (a spherical quadrilateral having four equal sides and four equal angles) is 73° 41′, find the angle and length of a diagonal.

Ans. Angle = 118° 04.5′, Diagonal 106° 16′.

7. The side of a regular spherical polygon (a spherical polygon having n equal sides and n equal angles) is a. Find the angle A of the polygon, the perpendicular p from the center of the polygon to one of the sides, and the distance r from the center to one of the vertices of the polygon.

Ans. $\sin \frac{1}{2} A = \dfrac{\cos(\pi/n)}{\cos \frac{1}{2} a}$, $\sin p = \tan \frac{1}{2} a \cot(\pi/n)$, $\sin r = \dfrac{\sin \frac{1}{2} a}{\sin(\pi/n)}$.

8. Find the perimeter of the polygon (Problem 7) when $p = 90°$.

Ans. 2π.

9. Compute the dihedral angles of a regular tetrahedron. Of a regular dodecahedron. *Ans.* 70° 31′ 44″, 116° 33′ 54″.

Suggestion. With a vertex of the polyhedron as a center describe a sphere. The points in which the three edges proceeding from the vertex intersect the sphere determine an equilateral spherical triangle the sides of which are known.

10. Compute the dihedral angles of a regular octahedron. Of a regular icosahedron. *Ans.* 109° 28′ 16″, 138° 11′ 23″.

Exercise 17

Oblique Spherical Triangles

1. The three face angles of a trihedral angle are BOC = 84° 24′, COA = 72° 18′, AOB = 60° 18′. Find the dihedral angles.

Ans. OA = 93° 40′, OB = 72° 48′, OC = 60° 36′.

2. Two planes intersect at an angle of 58° 40′. From a point of their line of intersection two lines are drawn, one in each plane,

making the angles $42^\circ 30'$ and $64^\circ 24'$ with the line of intersection. Find the angle which the lines thus drawn make with each other.

Ans. $50^\circ 33'$.

3. The great pyramid of Gizeh has a square for its base, and the angle between two edges at the vertex measures $96^\circ 01.2'$. Find the angle which each face makes with the horizon. *Ans.* $51^\circ 51'$.

4. A ten-sided pavilion is covered by a pyramidal roof. Two consecutive hips of the roof make an angle of 30°. Find the angle between two consecutive faces of the roof. *Ans.* $159^\circ 53'$.

5. The opposite faces of an obelisk are inclined at an angle of 16°. Find the face angles at the base of the obelisk and the angle between two adjacent faces. *Ans.* $82^\circ 04.6'$, $91^\circ 06.6'$.

6. The ridges of two gable roofs meet at right angles. The slope of each roof is 60°. Find the angle between the planes of the two roofs, and the angle the valley makes with each ridge.

Ans. $104^\circ 26.6'$, $63^\circ 26.1'$.

7. A mason cuts a stone in the shape of a pyramid with a regular hexagonal base. The edges are inclined at an angle of 30° with the base. Find the angle between two adjacent lateral faces, and the inclination of the faces to the base.

Ans. $149^\circ 18.6'$, $39^\circ 13.9'$.

8. If α, β, γ are the arcs joining any point in a trirectangular triangle to the vertices of the triangle, show that

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

9. An oblique parallelopiped has the three edges $OA = 2.59$, $AB = 3.65$, $OC = 7.21$, and the angles $AOB = 72^\circ 16'$, $BOC = 80^\circ 44'$, $COA = 41^\circ 18'$. Find its volume. *Ans.* 21.30.

46. Application to Geography and Navigation.

EXERCISE 18

1. Find the shortest distance measured along a great circle between New York, lat. $40^\circ 42' 44''$ N., long. $74^\circ 00' 24''$ W., and San Francisco, lat. $37^\circ 47' 55''$ N., long. $122^\circ 24' 32''$ W., the earth being considered a perfect sphere, radius 3959 miles. *Ans.* 2564 miles.

2. Find the area of a spherical triangle on the earth's surface ($r = 3959$ miles) whose spherical excess is 1°.

Ans. 273,575 square miles.

3. Compare the shortest distances in degrees of San Francisco, lat. 37° 47′ 55″ N., long. 122° 24′ 32″ W., and Seattle, lat. 47° 35′ 54″ N., long. 122° 19′ 59″ W., from Tokio, lat. 35° 39′ 18″ N., long. 139° 44′ 30″ E.

4. Find the distance in degrees and the bearing of Rio Janeiro, lat. 22° 55′ S., long. 43° 09′ W., from Cape of Good Hope, lat. 34° 22′ S., long. 18° 30′ E.

Ans. Distance 54° 29′, Bearing S. 84° 45′ W.

5. Find the first and final courses from San Francisco, lat. 37° 47′ 55″ N., long. 122° 24′ 32″ W., to Yokohama, lat. 35° 26′ 52″ N., long. 139° 38′ 41″ E. *Ans.* N. 56° 51′ W., S. 54° 17′ W.

6. A ship sails on an arc of a great circle a distance of 4150 miles from lat. 17° N., long. 130° W., the initial course being S. 54° 20′ W. Taking 1° = $69\frac{1}{6}$ miles, what is the latitude and longitude of its final position. *Ans.* Lat. 19° 41′ S., long. 178° 21′ W.

7. A vessel sails from Boston, lat. 42° 21′ N., long. 71° 03′ W., to Cape Town, lat. 33° 56′ S., long. 18° 28′ E. Find at what longitude the ship crosses the Equator and its course at this point.

Ans. Long. 17° 48′ W., course S. 41° 19′ E.

8. Find the distance at which a vessel sailing from Seattle to Tokio will cross the 180th meridian and its latitude at the time of crossing. (See Problem 3.)

9. Find the latitude and longitude of the place where a ship sailing from Cape of Good Hope to Rio Janeiro crosses the meridian at right angles. (See Problem 4.)

Ans. Lat. 34° 43′ S., long. 9° 15′ E.

10. Find the longitude and latitude of the place where a ship sailing from San Francisco to Yokohama crosses the meridian at right angles. (See Problem 5.)

11. The continent of Asia has nearly the shape of an equilateral triangle, each side being approximately 5500 miles. Find the area of the triangle (*a*) regarded as a plane triangle, (*b*) regarded as a spherical triangle, the radius of the earth being assumed 3960 miles.

Ans. 13,098,500 square miles; 17,228,400 square miles.

47. Applications from Astronomy.

EXERCISE 19

(For definitions of terms consult any dictionary or textbook on astronomy.)

1. How many seconds does it take for a star whose declination is $+64° 04'$ to cross the field of a telescope, the diameter of the field being $36'$? *Ans.* 329 seconds.

2. Find the approximate time of sunrise in Seattle, lat. $+47° 39'$, on Jan. 15, 1913. Suggestion. Look up the sun's declination.
Ans. $7^h\ 40.5^m$ A.M. local *apparent* time.

3. Find the length of the longest day at Seattle, lat. $+47° 39'$. Suggestion. When the sun is at its summer solstice its declination is $23° 27'$.

4. The moon's most northerly declination during this Saros occurred on March 19, 1913, and was $28° 44' 10''$. Find approximately how long it was below the horizon at San Francisco, lat. $37° 48' 24''$. *Ans.* $8^h\ 56^m$.

5. The zenith distance of the sun was observed to be $45° 26'$ the afternoon of a day when its declination was $+20° 32'$. If the latitude of the place was $+37° 10'$, what was the local apparent time?

6. The azimuth of the sun was measured and found to be $10° 14.2'$ and its zenith distance $25° 12.1'$ at a time when its declination was $+21° 39.2'$, find the latitude of the place.
Ans. $46° 34.1'$.

7. In Problem 6 find the local apparent time.
Ans. $0^h\ 30^m\ 13^s$.

8. At $1^h\ 15^m\ 16.1^s$ local apparent time the altitude of the sun was found to be $68° 21' 46''$ at a time when its declination was $+22° 41' 30''$. Find the latitude of the place.

9. In Problem 8 find the azimuth of the sun.

10. The altitude of the sun was measured and found to be $40° 18' 25''$ at a place whose latitude is $47° 39' 06''$ at $2^h\ 10^m\ 17.8^s$ local apparent time. Find the sun's declination.
Ans. $+ 6° 25' 53''$.

11. The northeastern end of the canal Phison on Mars is in Martian latitude $0° 03'$ N. and longitude $335° 10'$ and the southwestern end

is in latitude 40° 08′ S. and longitude 296° 58′. Find the length of Phison, the diameter of Mars being 4200 miles.

Ans. 1946.6 miles.

12. The declination of Algol is +40° 37′; find the azimuth of the star when setting at Ann Arbor, lat. +42° 17′.

13. The declination of Aldebaran is +16° 20.1′; find the azimuth of the star when setting at Seattle, lat. 47° 39.1′.

Ans. 114° 40.8′.

14. The declination of Procyon is +5° 26′ 55″; find the azimuth of the star when setting at Chicago, lat. 41° 50′ 01″.

15. The declination of 43H Cephei is now (1913) 85° 47′. Find its azimuth at Washington, D. C., lat. 38° 54′, 3^h 10^m after its meridian passage.

16. The declination of Polaris is now (1913) 88° 50′ 38″. Find its azimuth at Seattle, lat. 47° 39′ 06″, 5^h 01^m 20^s after its meridian passage. *Ans.* 178° 19′ 50″.

17. The right ascension and declination of Regulus are $\alpha = 10^h\ 03^m\ 44.4^s$, $\delta = +\ 12° \ 23'\ 34''$. On May 13, 1913, the moon's right ascension and declination were $\alpha = 9^h\ 58^m\ 37.3^s$, $\delta = +15°\ 32'\ 44''$. Find the angular distance between the moon's center and Regulus. *Ans.* 3° 23′ 20″.

18. The obliquity of the ecliptic is now (1913) 23° 27′ 02″. Find the celestial latitude and longitude of a star for which $\alpha = 3^h\ 15^m\ 20^s$, $\delta = +\ 36°\ 17'\ 56''$.

Ans. $\beta = +\ 17°\ 33'\ 19.7''$, $\lambda = 56°\ 11'\ 24.5''$.

19. What is the greatest altitude of a star on the equator in the meridian of Washington, lat. +38° 53′ 39″? *Ans.* 51° 06′ 21″.